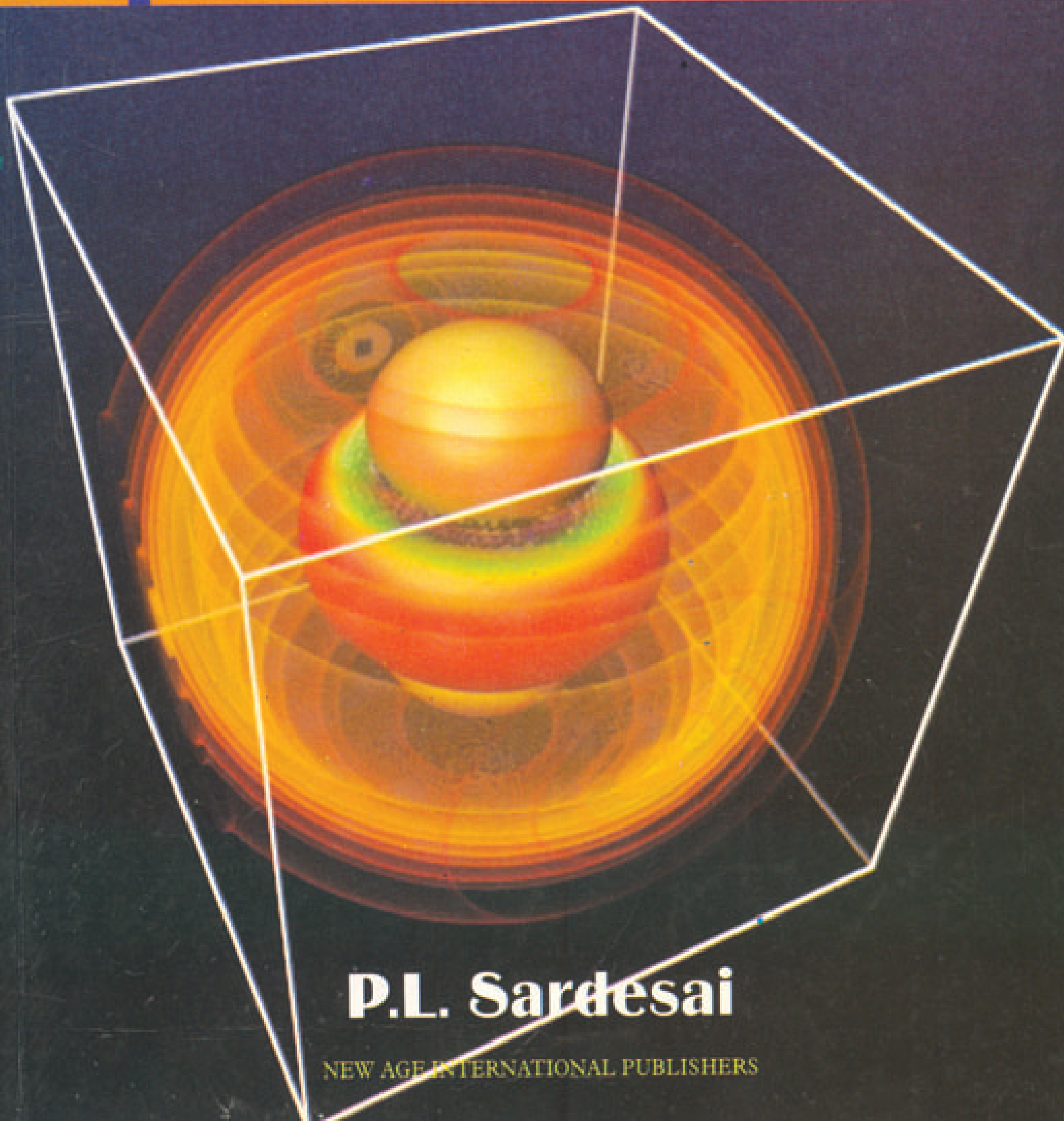
A Primer of Special Relativity



P.L. Sardesai

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This book is dedicated with love and reverence
to

Prof. V. M. Palekar & Prof. M. R. Dandavate
two absolutely great teachers

by
a relatively insignificant student

P.L. Sardesai

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Preface

The special theory of relativity is an essential part of the syllabus for B.Sc. (Physics) students in Indian Universities. Though many excellent books on the subject are available, most of our students find them to be high-brow because they are written by foreign authors with their own students in view as potential users. An average Indian student often finds that though he has understood the theory, he cannot solve the problems intended to test his grasp of the theory. This often results in loss of time for the student when he tries to refer to a number of books one after another to gain confidence in problem solving.

This book is specifically written for a student who needs to be led quickly and yet with sympathy along a path suitable for him so that he obtains a proficiency in the subject and develops the necessary skills and confidence to solve the problems based on the theory. Every effort is made to ensure that an average student masters the elements of the theory of special relativity in the shortest possible time without recourse to any other book.

Syllabi in special theory of relativity from a number of Indian Universities have been consulted for planning of this book. This book not only covers the syllabus recently suggested by the U.G.C. for B.Sc. General and Honours courses in Indian Universities but it also provides latitude for additional topics to suit individual taste. The structure of the book is such that it can be used as a textbook to suit individual taste, or to suit different courses as per the time available for its completion. The style is so facile that if necessary any topic can be assigned for self-study by the students. Besides the courses recommended by the U.G.C., the book can be used as a textbook for courses based on (i) first six chapters with some selected topics from Chapter 7; (ii) first seven chapters; (iii) first eight chapters etc.

Throughout the writing of the book, the author has kept reminding himself of his own student days—he has in view an average student putting in sincere efforts to get a clear understanding of elements of the special theory of relativity. The author would be personally obliged to receive suggestions to make this book more useful to our student community.

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Chapter 1

Pre-Einstein Relativity

INTRODUCTION

The theory of relativity deals with relations which exist between physical quantities (such as mass of a particle, length of a rod, electric field at a point etc.) as they appear to different observers in *relative* motion. The observers considered in this book are restricted to those in inertial frames of reference. The theory is then called *restricted theory* or *Special Theory of Relativity* (STR). When no such restriction is made, the theory is called the *General Theory of Relativity*.

1.1 INERTIAL FRAMES OF REFERENCE

Measurements of time and distance are carried out in a definite coordinate system (such as *OXYZ* of Fig. 1.1); the system in which the clock and the metre scale used for the measurement are at rest. The coordinate system so used is called a *frame of reference*.

Of all possible frames of reference, we are concerned only with the so called *inertial frames of reference* for the purpose of our study in this book.

Inertial frame of reference is that in which the law of inertia (Newton's first law of motion) holds, that is a frame in which a body that is acted upon by zero net external force moves with a constant velocity. The law of inertia holds in any frame of reference, which happens to move with a constant velocity relative to a given inertial frame. Therefore, any frame of reference, which moves with a constant velocity relative to an inertial frame, is also an inertial frame.

Newton believed that a frame of reference fixed with respect to the stars is an inertial frame. A space ship drifting in outer space without spinning and with its engines shut off would be an ideal inertial frame. However for all practical purposes, any frame of reference fixed to the earth such as a railway station or a laboratory can be taken as an inertial frame. Thus a railway station is an inertial frame and a train travelling at constant velocity with respect to the railway station is also an inertial frame.

1.2 THE PRINCIPLE OF RELATIVITY (GALILEAN INVARIANCE)

The Principle of Relativity (PR) applies to inertial frames of reference. This principle states that *the laws of Physics take the same mathematical form in all inertial frames.*

The above principle is sometimes stated in a different way viz. the basic laws of Physics are identical in all frames of reference which are moving with uniform velocity (unaccelerated) relative to one another. This is called as *Galilean Invariance*.

Yet another way of stating the principle is: It is impossible by using any physical law to distinguish between inertial frames.

1.3 GALILEAN TRANSFORMATIONS (G.T.)

A coordinate system also called a reference frame, consists of three rectangular axes to indicate the position (x, y, z) in that frame as well as a clock (not shown) attached to the frame to indicate time (t).

Figure 1.1 shows two inertial frames S and S'. Frame S' is moving with a constant speed v in the X-direction with respect to frame S.

We suppose that the clocks at rest in S and S' are synchronized and set to zero at the instant the origins O and O' of S and S' coincided.

Our aim is to find equations which relate the time and space coordinates of a body in the frames S and S' which are in relative motion.

As seen by an observer at rest in S, origin O' of S' at time t is located at the point (vt, o, o).

Suppose a particle is present at point P. Let the coordinates of P be denoted by (x, y, z) in frame S and by (x', y', z') in frame S'. As seen from Fig. 1.1, these coordinates are related by

$$x' = x - vt;\ y' = y \text{ and } z' = z \qquad \text{... (1a)}$$

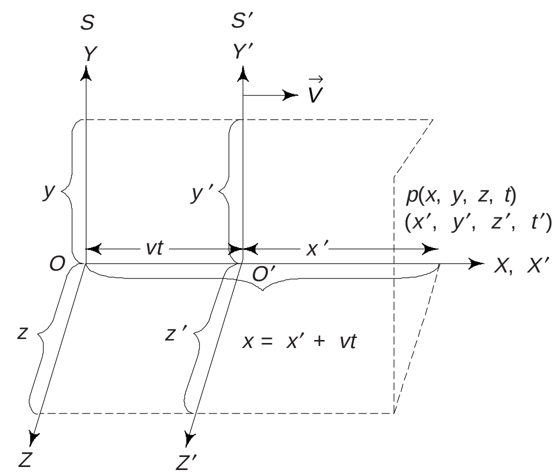

Fig. 1.1 Galilean Transformations

It was assumed by earlier physicists that clocks in S and S' when synchronized initially will continue to run at the same rate so that the two observers will read the same time. This leads to

$$t' = t \qquad \text{... (1b)}$$

The transformation equations which relate the time and space coordinates in frames S and S' are called Galilean Transformations (G.T.) viz.

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \qquad \text{... (1)}$$

Note carefully that according to G.T.; (i) *the concept of time is absolute* $(t' = t)$ and (ii) *the concept of space* that is the concept of distance or length *is also absolute* $(L' = L$ as shown in Illustrative Example 1 of this chapter).

1.4 APPLICATION OF G.T. TO MECHANICS

On the basis of G.T., it is possible to obtain relations between physical quantities measured by two inertial observers in relative motion. Some of these are merely listed below:

(a) If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then

$$\vec{u}' = \vec{u} - \vec{v} \qquad \text{... (1)}$$

where $\vec{v}$ is the velocity of S' relative S. This is the familiar 'common sense' formula of relative velocity!

(b) Acceleration of a particle as measured in S and S' is the same. That is say

$$\vec{a}' = \vec{a} \qquad \text{... (2)}$$

(c) The mass of a particle has the same value in different inertial frames. If m' and m are the masses of a particle as determined in frames S' and S respectively, then (see Illustrative Example 2 of this chapter).

$$m' = m \qquad \text{... (3)}$$

Hence equation of motion such as $\vec{F} = m\vec{a}$ in frame S is transformed into $\vec{F}' = m\vec{a}'$ in frame S'. Not only this equation but in fact Newtonian Mechanics has the same form in different inertial frames according to pre-Einstein relativity. So far so good!

1.5 APPLICATIONS OF G.T. TO ELECTROMAGNETISM

Fundamental laws of Electromagnetism can be expressed in a very elegant set of mathematical equations called Maxwell's Equations (M.E.). From these equations Maxwell deduced that electromagnetic waves (light, radio waves etc.) travel in empty space (and for all practical purposes through air) with a constant speed.

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s}$$

where ε_0 = permitivity of free space

μ_0 = permeability of free space.

On application of G.T., it is found that the form of M.E. changes, that is *M.E. are not form invariant* as required by the Principle of Relativity. This can be seen in a different but easier way by using the idea of relative velocity.

If $\vec{c}$ is the velocity of a light pulse as measured by an observer in S, then the velocity of the same light pulse as measured by an observer in S' is by Eqn. (1) of Art. 1.4

$$\vec{c}' = \vec{c} - \vec{v} \qquad \text{... (1)}$$

where $\vec{v}$ is the velocity of S' relative to S.

It is obvious that magnitude of $\vec{c}'$ will in general be different from that of $\vec{c}$. Clearly all is not well with Physics!

Physicists were then inclined to believe that the speed of light in empty space was constant (3×10^8 m/s) in a certain *privileged* inertial frame and that it was different in all other inertial frames.We shall study more about this in the next chapter.

SUMMARY

Special Theory of Relativity deals with relations which exist between physical quantities as they appear to different inertial observers.

Inertial frames of reference are those in which the law of inertia holds.

Principle of Relativity requires that the laws of Physics take the same mathematical form. G.T. are

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

In pre-Einstein relativity (G.T.), the concept of time is absolute, the concept of space is also absolute.

Under G.T., Newtonian mechanics is form invariant but the theory of electromagnetism (Maxwell's Equations) is *NOT.*

ILLUSTRATIVE EXAMPLES

Example 1 Use G.T. to show that the distance measured is independent of the frame of reference.

Solution Let (x_1', y_1', z_1') and (x_2', y_2', z_2') be the coordinates of some two points P_1 and P_2 respectively at some instant of time t' (= t) as observed in S'. (P_1 and P_2 could be the end points of a rod.

The distance between P_1 and P_2 as measured in S' is

$$L' = [(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2]^{1/2}$$

$$= [\{(x_2 - vt) - (x_1 - vt)\}^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

by G.T. See Eqns. (1) of Art. 1.3.

$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} = L \text{ say}$$

where L = distance as measured in frame S.

The distance is thus independent of the frame of reference.

Example 2 Show that mass is invariant relative to Galilean Transformations (G.T.) between inertial frames.

Solution Consider a collision of two particles in an inertial frame S. Let $\vec{u}_1$ and $\vec{u}_2$ denote the velocities of the two particles of mass m_1 and m_2 before the collision. Let $\vec{v}_1$ and $\vec{v}_2$ denote velocities after the collision. Then from conservation of linear momentum,

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2$$

$$\therefore \quad \vec{u}_1 + \frac{m_2}{m_1} \vec{u}_2 = \vec{v}_1 + \frac{m_2}{m_1} \vec{v}_2$$

$$\therefore \quad \left(\frac{m_2}{m_1}\right)(\vec{u}_2 - \vec{v}_2) = \vec{v}_1 - \vec{u}_1$$

$$\therefore \quad \frac{m_2}{m_1} = \frac{|\vec{v}_1 - \vec{u}_1|}{|\vec{u}_2 - \vec{v}_2|} \qquad \text{... (1)}$$

This eqn. allows the ratio of masses to be determined from collision experiments. If one mass is chosen as unit, the mass of the other particle can be determined. Thus eqn. (1) serves to provide a measurement of mass.

Consider the same collision as observed in frame S'. If $m_1', m_2', \vec{u}_1', \vec{u}_2', \vec{v}_1', \vec{v}_2'$ are the corresponding quantities in S', then

$$\frac{m_2}{m_1} = \frac{|\vec{v}_1 - \vec{u}_1|}{|\vec{u}_2 - \vec{v}_2|} = \frac{m_2'}{m_1'} = \frac{|\vec{v}_1' - \vec{u}_1'|}{|\vec{u}_2' - \vec{v}_2'|} \qquad \text{... (2)}$$

since the ratio of masses must be the same in either frame. (otherwise no trader could carry on his business using balances!).

If the first particle is taken to be of unit mass in either frame, then $m_1 = m_1' = 1$ by choice of unit mass. $\therefore\ m_2' = m_2$ from eqn. (2). That is the mass of a particle has the same value in all inertial frames.

EXERCISES

1. What do you mean by a frame of reference?
2. What are inertial frames? Give examples.
3. State and explain the Principle of Relativity.
4. Obtain Galilean equations of transformation for two inertial frames in relative motion.
5. Comment on the nature of time and space in G.T.
6. On the basis of G.T. show that the force acting on a particle is independent of the inertial frame in which it is measured.

   Hint: Differentiate eqn. $\vec{u}' = \vec{u} - \vec{v}$ with respect to time get, $\vec{a}' = \vec{a}$. Note that $\vec{v}$ is constant. Also $m' = m$ from Illustrative Example 2. Use $\vec{F} = m\vec{a}$ to show $\vec{F}' = \vec{F}$.
7. What is the difference in the outcome of applying G.T. to mechanics and to electromagnetism? Explain.

Suggested Further Reading

Sherwin C.W.: *Basic Concepts of Physics* (Holt, New York 1961).
Einstein A. and Infeld L.: *The Evolution of Physics* (Simon and Schuster New York)
H. Bondi, *"The Teaching of Special Relativity"*, Physics Education, 1,223 (1966).
Robert Resnick: *Introduction To Special Relativity* (John Wiley and Sons).

Chapter 2

Ether, The Myth and The Truth

INTRODUCTION

From his electromagnetic theory, Maxwell showed that electromagnetic waves travel in vacuum with a constant speed $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. Surprising but true, the derivation of this result makes no reference to the speed of the source emitting the electromagnetic waves. This is of great significance but the early physicists failed to recognize it. They were led astray because they did not know the true nature of light. Some of the work carried out by them in connection with search for the medium for light propagation, motion of earth relative to this medium etc. is described in this chapter.

2.1 THE ETHER

Sound waves need a medium to carry them, so by analogy it was believed that a medium with mechanical properties (density, elasticity etc.) must exist to transmit light waves. Because light waves can be transmitted through vacuum, it was believed that a medium called *ether* filled all space. That is, ether was supposed to be present in vacuum, in the space between atoms, molecules and so on.

Ether was thought to be the *softest* of all substances because matter can move through it without any friction or resistance. Earth and other planets travel through this medium year after year without any reduction in speed.

Light travels through ether at such a fantastically high speed that extremely strong restoring forces are set up in ether when it is disturbed by propagation of light. This requires that elastic constants of ether must be the highest of all materials. In other words, ether must be the *hardest* of all materials.

The hypothesis of ether is clearly self-contradictory and absurd. But the scientists did not give it up because they thought a wave cannot travel without a mechanical medium.

By analogy with sound waves, scientists believed that the speed of light c as derived by Maxwell must be the velocity of light with respect to ether. According to G.T., the speed of light would then be different in different inertial frames in motion relative to ether. Ether is thus a privileged frame and the Principle of Relativity does not apply to Maxwell's Equations, that is to the theory of Electromagnetism developed by Maxwell.

2.2 THE CRUX

Since the theory of Electromagnetism is as fundamental as Newtonian Mechanics, the physicists faced a very serious problem. They had three possibilities to choose from. (i) Electromagnetic theory (Maxwell's Equations etc.) is wrong and should be given up. (ii) The Principle of Relativity should be given up. (iii) G.T. (Concept of absolute time, absolute space etc.) are incorrect and should be given up. The choice is extremely hard to make.

The choice was finally made by Einstein. However, before we come to that, some landmark experiments in the developments of the subjects will be taken up.

2.3 MICHELSON-MORLEY EXPERIMENT (1887)

If ether exists it should be possible to detect the motion of the earth through the ether and in particular to determine the speed of earth (v) relative to ether. This is what Michelson and Morley tried to find out using an instrument called Michelson's interferometer.

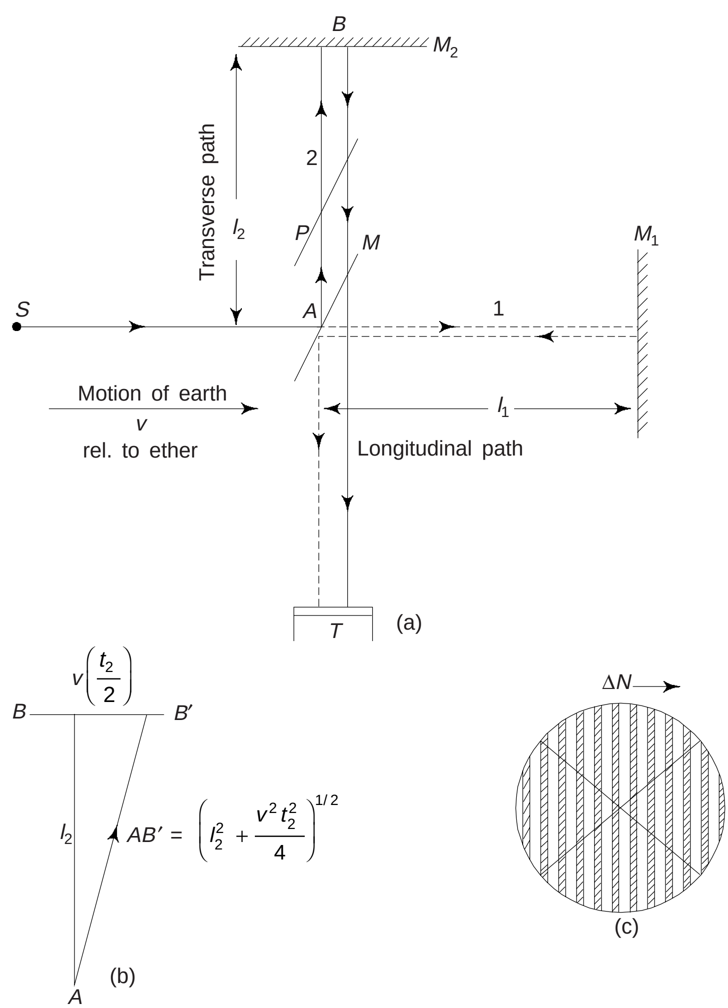

Fig. 2.1 Michelson-Morley Experiment

Light from a monochromatic source S is spilt into two beams by a lightly silvered mirror M so that the transmitted beam (beam 1) and the reflected beam (beam 2) are at right angles and have almost equal intensities.

Beam 1 shown dotted (along the longitudinal path, that is along the earth's motion) is reflected from mirror M_1 and enters the telescope T after striking the mirror M.

Beam 2 (along the transverse path, that is perpendicular to earth's motion) is reflected from mirror M_2 and enters the telescope after passing through mirror M. Glass plate P is inserted in the transverse path so that both the beams travel through the same thickness of glass.

Beams 1 and 2 produce a steady interference pattern which can be seen through the telescope T. See Fig. 2.1 (c).

Because of motion of earth along SAM_1, beam 1 travels with speed $(c - v)$ towards M_1 and with speed $(c + v)$ in the reverse direction. Hence the total time of travel for the longitudinal path is

$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c-v} = \frac{2l_1 c}{c^2 - v^2} = \left(\frac{2l_1}{c}\right)\left(\frac{1}{1-\frac{v^2}{c^2}}\right) \qquad \text{... (1)}$$

Due to motion of the interferometer along with the earth, beam 2 strikes the mirror M_2 not at B but at B′. See Fig. 2.1 (b).

If $\frac{t_2}{2}$ is the time taken by beam 2 to reach M_2 after being reflected from M, then $BB' = \frac{vt_2}{2}$, where v is the velocity of earth as indicated in Fig. 2.1 (a). The path of light is AB' and is seen to be given by

$$AB' = \frac{ct_2}{2} = \left(l_2^2 + \frac{v^2 t_2^2}{4}\right)^{1/2}$$

$$\therefore \qquad \frac{c^2 t_2^2}{4} = l_2^2 + \frac{v^2 t_2^2}{4}$$

$$\therefore \qquad \frac{t_2^2}{4} = \left(\frac{l_2}{c}\right)^2 \left(\frac{1}{1-v^2/c^2}\right)$$

$$\therefore \qquad \frac{t_2}{2} = \left(\frac{l_2}{c}\right)\left(\frac{1}{1-v^2/c^2}\right)^{1/2}$$

The total time of travel for the to and fro journey (M to M_2 and back to M) is

$$2\left(\frac{t_2}{2}\right) = \left(\frac{2l_2}{c}\right)\frac{1}{(1-v^2/c^2)^{1/2}} \qquad \text{... (2)}$$

The time difference between the longitudinal and transverse journeys is

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left[\frac{l_2}{(1-v^2/c^2)^{1/2}} - \frac{l_1}{(1-v^2/c^2)}\right] \qquad \text{... (3)}$$

and the path difference (p.d.) between the interfering beams is $\Delta L = c\Delta t$.

The steady interference pattern mentioned above is due to this path difference.

When the entire experimental set up (kept floating in a mercury tank) is rotated through a right angle, the roles of the longitudinal and transverse paths are interchanged. The time for the longitudinal path is now

$$\therefore \qquad t_2' = \left(\frac{2l_2}{c}\right)\left(\frac{1}{1-v^2/c^2}\right) \qquad \text{... (4)}$$

and for the transverse path, it is

$$t_1' = \left(\frac{2l_1}{c}\right)\frac{1}{(1-v^2/c^2)^{1/2}} \qquad \text{... (5)}$$

The path difference is $\Delta L' = c(t_2' - t_1') = c\Delta t'$.

The change in path difference is

$$\Delta L' - \Delta L = 2\left[\frac{l_2}{1-v^2/c^2} - \frac{l_1}{(1-v^2/c^2)^{1/2}} - \frac{l_2}{(1-v^2/c^2)^{1/2}} + \frac{l_1}{1-v^2/c^2}\right]$$

$$= 2\left[\frac{l_1+l_2}{1-v^2/c^2} - \frac{l_1+l_2}{(1-v^2/c^2)^{1/2}}\right]$$

$$= 2(l_1 + l_2)\left\lfloor(1 - v^2/c^2)^{-1} - (1 - v^2/c^2)^{-1/2}\right\rfloor$$

Since $\frac{v}{c} \ll 1$, we can use the binomial approximation

$$(1 - x)^n \approx 1 - nx \text{ when } x \ll 1.$$

$$\Delta L' - \Delta L = 2(l_1 + l_2)\left[\left(1+\frac{v^2}{c^2}\right) - \left(1+\frac{v^2}{2c^2}\right)\right]$$

$$= 2\,(l_1 + l_2)\,\frac{v^2}{2c^2} = \frac{v^2}{c^2}\,(l_1 + l_2) \qquad \text{... (6)}$$

If the path difference changes by one wavelength (λ) in the course of rotation, then there should be a shift of one fringe across the cross-wire. Hence due to rotation of the apparatus through a right angle, the number of fringes shifting across the cross-wire is expected to be

$$\Delta N = \frac{\Delta L' - \Delta L}{\lambda} = \frac{1}{\lambda}\,(l_1 + l_2)\left(\frac{v^2}{c^2}\right) \qquad \text{... (7)}$$

The experiment consists of rotating the apparatus through a right angle and counting the number of fringes ΔN passing the cross-wire in the course of rotation of the apparatus. Experimentally it was found that *there was no fringe shift whatsoever in the course of rotation.*

At first Michelson and Morley considered the possibility of motion of the entire solar system in such a direction and of such a magnitude so as to exactly cancel the velocity of earth relative to ether (that is $v = 0$) at the time of the experiment. In that case, the earth would reverse its velocity after six months so that the velocity of the solar system and that of the earth would add instead of canceling. However, repetition of the experiment six months later once again gave a null result, that is no fringe shift was observed. In fact Michelson and Morley made observations day and night and at all seasons of the year but could not detect any fringe shift. Similar experiments were undertaken by many observers but fringe shift could never be observed in spite of refinements in the apparatus. See Illustrative Example 1 of this chapter to get a quantitative feel of the experiment.

2.4 ATTEMPTS TO EXPLAIN NULL RESULT

A number of attempts were made to explain the null result of Michelson-Morley experiment. All such attempts presumed the correctness of G.T. and turned out to be utter failures. Some of them are described here to point out their futility.

[A] Ether Drag Hypothesis

The speed of light is constant equal to c with respect to ether; but according to this hypothesis, a body moving through the ether drags or carries the ether in its neighbourhood alongwith it.

If the earth drags the ether in its neighbourhood (just as a train carries the air along with it) then Michelson's interferometer is at rest with respect to ether; that is $v = 0$ and hence $\Delta N = 0$ is Michelson-Morley experiment.

Ether drag hypothesis thus explains the null result of Michelson-Morley experiment and at the same time it retains the privileged either frame.

The above explanation is contradicted by the phenomenon of aberration. *Aberration* is variation in the apparent position of a star due to the motion of the observer along with the earth.

To see a fixed star from the earth with a telescope, one cannot point it in the exact geometrical direction where the star lies but a little forward in the direction of the earth's motion around the sun. Then only the star can be sighted. Fig. 2.2 (a) shows the case of a star whose true position is directly overhead.

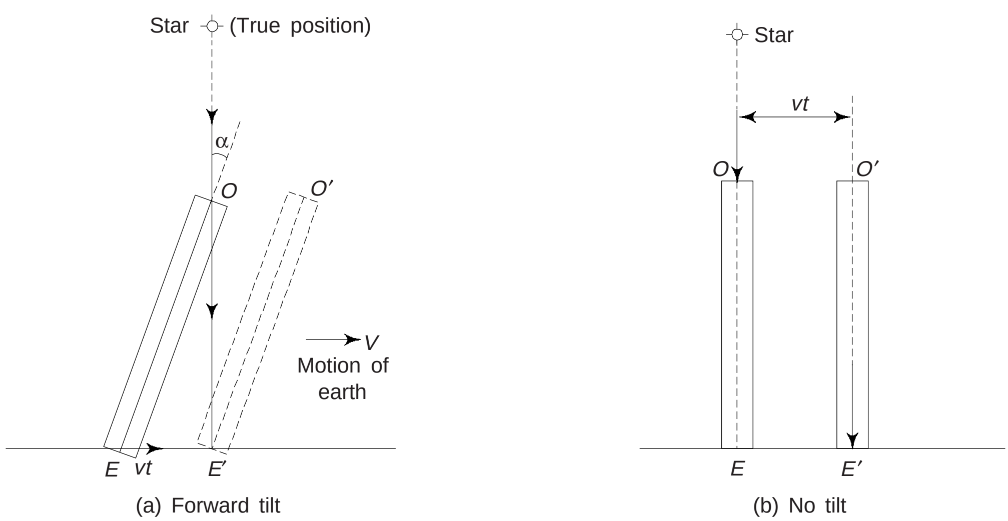

Fig. 2.2 Aberration

Starlight enters the center of the objective O of the telescope. It will then reach the center of the eyepiece E only if the telescope is tilted through an angle α to the vertical such that in the same time t, the light takes to travel $OE' = ct$, the telescope eyepiece travels from E to E'. If v is the velocity of the earth in its orbit then $EE' = vt$. Hence

$$\tan \alpha \approx \frac{EE'}{OE'} = \frac{vt}{ct} = \frac{v}{c}$$

Since $v \ll c$, angle α is very small. We may therefore write

$$\tan \alpha \approx \alpha = \frac{v}{c} \qquad \text{... (1)}$$

After six months, the earth is on the opposite side of its orbit and the direction of v is reversed. The telescope must now be tilted backwards through the same angle α if the star is to be viewed through the telescope.

If the ether drag hypothesis is correct, then the earth should drag the ether and hence the light along with it. See Fig. 2.2(b). The light beam and the telescope would both travel along with the earth. There is then no need to tilt the telescope forward or backward. In other words, the phenomenon of aberration would not be observed. Ether drag hypothesis is therefore ruled out by aberration.

[B] Elastic Corpuscles Hypothesis

According to this hypothesis, light consists of extremely small elastic corpuscles emitted with speed c relative to the source of light. The velocity of these corpuscles is assumed to be independent of the state of motion of the medium (such as ether) transmitting the light. The light corpuscles are reflected from mirrors as per laws of reflection of elastic particles. By using pre-Einstein relativity (G.T.) it can be shown that speeds of such corpuscles of light would be the same along longitudinal and transverse paths in the Michelson-Morley experiment. The null result of the experiment is thus explained.

However there is a very fundamental objection to the above explanation. Maxwell's theory describes light clearly as a wave propagation and not as a motion of mechanical particles or corpuscles with speed c.

Moreover, if the above hypothesis is correct, then it should cause some peculiar effects in the light reaching us from some binary stars. These are pairs of stars rapidly rotating around their centre of mass. Because of rapid motion of a star receding away and then approaching us in the course of rotation, it should be possible to see "*ghost stars*" viz. the same star at two places simultaneously. See Fig. 2.3

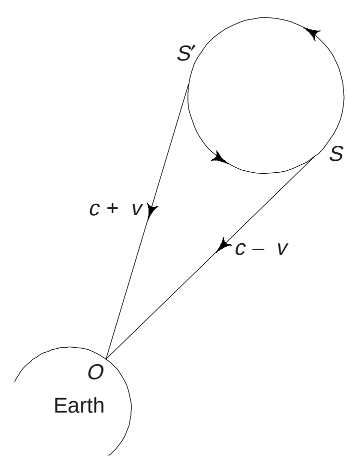

Fig. 2.3 Ghost Stars

Star in position S is receding away with speed v and light emitted by it would travel to earth along SO at speed $(c - v)$. (Similar to a bullet fired with speed c from a train receding away at speed v). When the same star arrives at S' after a time t, the star is approaching O with speed v. Light from S' now travels towards O with a speed $(c + v)$. If

$$t + \frac{S'O}{c + v} = \frac{SO}{c - v}$$

then an observer at O would see the same star at S and S' for a moment. Such a "ghost star" should therefore appear and disappear regularly in the course of motion of binary stars. This is not a mere fantasy. Binary stars with just the right speeds of motion and distances from the earth necessary for 'ghost formation' do exist. But no 'ghost stars' have ever been observed. The elastic corpuscular hypothesis is therefore ruled out.

[C] The Lorentz-Fitzgerald Contraction

The speed of light is constant equal to c with respect to ether. However according to this hypothesis all bodies are contracted in the direction of their motion relative to stationary ether by a factor of $\left(1 - \frac{v^2}{c^2}\right)^{1/2}$.

(Such a contraction was supposed to be brought about by the effect of motion on the electric forces between atoms of the body in motion). The predicted fringe shift in Michelson-Morley experiment then turns out to be

$$\Delta N = \frac{l_1 - l_2}{\lambda}\left(\frac{v^2}{c^2} - \frac{v'^2}{c'^2}\right) \quad \text{... (2)}$$

when the velocity of the interferometer with respect to ether changes from v to v'.

In the original experiments of Michelson and Morley, the distances l_1 and l_2 were almost equal and hence ΔN is nearly zero in accordance with the null result. However, Kennedy and Thorndike repeated the experiment using an interferometer with unequal arms ($l_1 \neq l_2$). Even with a large value for ($l_1 - l_2$), no fringe shift could be observed.

2.5 ENTER, EINSTEIN

While many physicists were busy concocting theories and explanations for the null result of Michelson-Morley experiment, it was the genius of Einstein that unravelled the deepest significance of Principle of Relativity applied to Electromagnetism. In all probability, Einstein was not fully aware of Michelson-Morley experiment and did not attempt to explain their null result. He was of the view that all the laws of Physics should have the same form in all inertial frames and saw that this is possible only if G.T. are replaced (choice number III) by a new set of equations in which neither space nor time is absolute.

Einstein realized that according to the Principle of Relativity, *the speed of light should be constant c in all directions with respect to all inertial observers.* The speed of light is then the same along longitudinal and transverse paths of Michelson interferometer and the null result of Michelson-Morley experiment is exactly as expected.

Constancy of speed of light relative to all inertial observers is immediately seen to be against our common sense concept of relative velocity which follows from Galilean Transformations. Einstein was convinced that G.T. based on the concept of absolute time and absolute distance are at fault and should be replaced by a new set of equations of transformation which must preserve constancy of speed of light and leave the laws of Electromagnetism invariant as required by the Principle of Relativity. In fact Einstein developed his theory on the twin requirements (i) invariance of the laws of Physics—that is the laws of Physics should have the same form in all inertial frames since all such frames are completely equivalent. (ii) constancy of the speed of light.

Einstein argued that it is impossible by means of any physical measurements to label a coordinate system as intrinsically ‘stationary’ or ‘uniformly moving’. One can only infer that two systems are moving relative to each other. Thus measurements or experiments carried out entirely within an inertial frame will be incapable of distinguishing that system from all others moving uniformly with respect to it. Such is the significance of ‘equivalence’ of all inertial frames.

The new transformation equations derived by Einstein are (for certain historical reasons) known as Lorentz Transformations (L.T.). They leave the laws of Electromagnetism invariant or unchanged in form as required by the Principle of Relativity. However, the requirement that the laws of mechanics should also be invariant under the transformations is seen to lead to a modification of Newtonian Mechanics. The changes in the mechanical theory which are required to make it invariant become apparent only at very high speeds.

2.6 STRANGE BEHAVIOUR OF LIGHT

We have said above that the speed of light is the same for all inertial observers. This behaviour of light (electromagnetic radiation) is very strange. It is altogether different from what our common sense tells us to expect. Moreover no other wave motion exhibits this strangeness. The strange behaviour of light is very strikingly seen from the outcome of a recent experiment.

In Fig. 2.4(a), L is a source which is at rest in the laboratory. Electromagnetic radiations emitted by the source is observed to travel at a speed $c = 3 \times 10^8$ m/s.

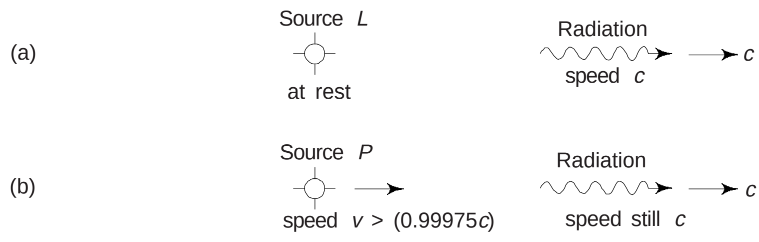

Fig. 2.4 Strange Behaviour of Light

P in Fig. 2.4 (b) is another source (pion) which is observed to travel at a speed greater than (0.99975)*c* with respect to the laboratory. Radiation emitted by this source *P* (in pion decay) is also observed to travel at a speed of c = 3×10^8 m/s. Is that not strange?

2.7 ETHER, THE MYTH AND THE TRUTH

The concept of ether with its self-contradictory properties is obviously absurd. We now know that light consists of oscillating electric and magnetic fields that can travel in vacuum or free space with speed c and does not need ether or any mechanical medium whatsoever for its propagation.

Ether is just a myth—in truth, what the scientists earlier called ether is just empty space.

It may be added here that the null result of Michelson-Morley experiment does not by itself disprove the existence of ether—the null results means that the speed of light is always the same in all directions and is independent of the relative uniform motion of the observer and the source.

In retrospect one may say, the concept of ether clouded the thinking of the scientists and a lot of scientific effort was wasted in retaining the concept. Scientific progress is not unimpeded!

SUMMARY

Scientists did not know the true nature of light. They assumed the existence of ether for the propagation of light. The properties attributed to ether were ridiculous and self-contradictory.

Physicists believed that light travels with speed *c* with respect to the ether medium and that velocity of light should be different for different observers in relative motion with respect to ether. Ether was considered to be a privileged frame in contradiction to the Principle of Relativity (P.R.).

Michelson and Morley conducted experiments to determine (v) the speed of earth relative to ether by using the idea of relative velocity applied to light travelling along (longitudinal path) and perpendicular (transverse path) to the earth's motion. They failed to detect any motion (null result).

Attempts to explain null result were futile. Hypothesis of ether drag explains the null result but is ruled out by phenomenon of aberration viz. variation in the apparent position of a star due to the motion of the observer with the earth.

Elastic corpuscular hypothesis can explain the null result but is ruled out by absence of 'ghost stars'—that is, the same star momentarily appearing to be present at two places.

Lorentz-Fitzgerald contraction explains the null result but his hypothesis is ruled out by experiments of Kennedy and Thorndike.

Einstein understood the true significance of P.R. according to which the speed of light should be the same for all inertial observers. He realized that G.T., concepts of absolute time and length, relative velocity etc. were at fault. He argued that G.T. should be replaced by another set of equations which maintained the constancy of the speed of light for different inertial observers. In fact, he based his theory of relativity on the twin postulates of constancy of speed of light and invariance of the laws of Physics in different inertial frames.

Experiments show that all inertial observers irrespective of their motion relative to the source emitting light, find that the speed of light is c. This may appear strange but it is true.

Light does not need any medium for propagation and the concept of ether is just a myth. In truth what the scientists thought was ether is just empty space. The null result of Michelson Morley experiments points to the constancy of speed of light.

ILLUSTRATIVE EXAMPLE

Example 1 In the Michelson-Morley experiment, $(l_1 + l_2)$ was 22 m and the wavelength of light used was 6000 A°. They assumed that ether is fixed relative to the sun so that the earth and the interferometer move through the ether at a velocity $v = 3 \times 10^4$ m/s which is the orbital speed of the earth about the sun. Calculate the fringe shift they expected to observe.

Solution

$$\lambda = 6000 \times 10^{-10} \text{ m} = 6 \times 10^{-7} \text{ m.}$$

$$\frac{v}{c} = \frac{3\times 10^4}{3\times 10^8} = 10^{-4}$$

$$\Delta N = \left(\frac{l_1 + l_2}{\lambda}\right)\left(\frac{v^2}{c^2}\right) = \frac{22}{6\times 10^{-7}} \; (10^{-4})^2 = 0.37$$

The experiment was sensitive enough to detect a shift as small as 0.01.

EXERCISES

1. Why was the concept of ether introduced? Why do you think the concept is absurd?
2. Describe in detail the experiments of Michelson and Morley on the velocity of earth relative to the ether frame.

   In their experiment Michelson and Morley managed to have $l_1 = l_2 = 11$ meters approximately. The wavelength of light used is about 5.5×10^{-7}m. The velocity of earth was taken to be about 30 km/s relative to ether assumed to be fixed relative to the sun. Estimate the fringe shift expected.

   (*Ans*. 0.4)
3. Discuss critically the different interpretations of the null result of Michelson-Morley experiment.
4. What is the significance of the null result of Michelson-Morley experiment? Does it disprove the existence of ether? Explain.
5. Derive Equation (2) of Art. 2.4 used by Kennedy and Thorndike.

   Hint:

$$t_1 = \frac{2l_1 (1 - v^2/c^2)^{1/2}}{c(1 - v^2/c^2)} = \frac{2l_1}{c(1 - v^2/c^2)^{1/2}}$$

$$t_2 = \frac{2l_2}{c(1 - v^2/c^2)^{1/2}}$$

   Assume the speed of earth to change from v to v'. Take $\dfrac{1}{(1 - v^2/c^2)^{1/2}} \approx \left(1 + \dfrac{v^2}{2c^2}\right)$

6. An air plane travels from A to B and back. The speed of the plane is c. If $AB = 1$, find the (a) time taken for the round trip; (b) time taken for the round trip when a wind of velocity $v \ll c$ blows from B to A; (c) time taken for the round trip when a wind of velocity $v \ll c$ blows from A to B; (d) time taken for the round trip when the wind is blowing perpendicular to AB with a velocity $v \ll c$.

$$\left(\textit{Ans.} \text{ (a) } \frac{2l}{c}; \text{ (b) } \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right); \text{ (c) } \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right); \text{ (d) } \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right) \right)$$

7. Elucidate Einstein's ideas on the principle of relativity.
8. What are the twin bases for Einsteinian relativity?
9. What is 'strange' about the speed of light propagation? How is it demonstrated experimentally?
10. Does light need any medium for its propagations? Explain.
11. Figure 2.5 shows the apparent change in the position of a fixed overhead star in the course of earth's motion around the sun. A cone of aberration of angular diameter $2\alpha = 41''$ is generated by the telescope viewing the overhead star during the year. See Fig. 2.5(b). Assume the earth goes round the sun along a circular path with constant speed v. Estimate this speed v.

(*Ans.* $v = 30$ km/s) $\left[\textbf{Hint: } \frac{v}{c} = \alpha \text{ (radians)} \right]$

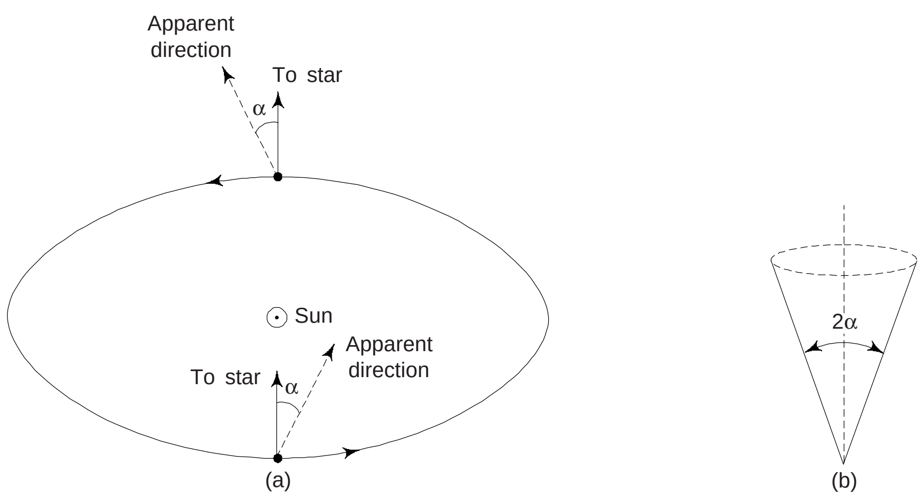

Fig. 2.5 For Exercise 11

12. Figure 2.6 shows one of the binary stars moving in a uniform circular motion with speed v and period T. In position (1) the star is moving away from the earth along the line connecting them and in position (2) the star is approaching the earth along the line connecting them. Assume the distance 1 shown in the figure to be very large and the positions (1) and (2) to be half-orbit apart.

    On the basis of elastic corpuscular hypothesis (a) show that the star appears to travel from position (1) to position (2) in a time $\frac{T}{2} - \frac{2lv}{c^2 - v^2}$ and from position (2) to position (1) in a time $\frac{T}{2} + \frac{2lv}{c^2 - v^2}$; (b) Show that the condition for ghost formation is $\frac{T}{2} = \frac{2lv}{c^2 - v^2}$.

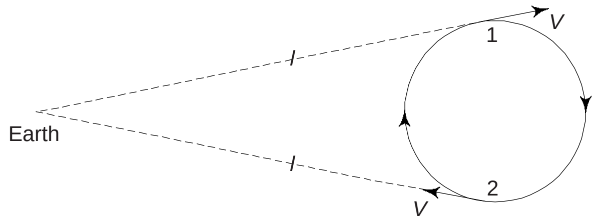

Fig. 2.6 For Exercise 12

[**Hint:** (a) $T_1 = t_1 + \dfrac{1}{c - v}$; $T_2 = t_1 + \dfrac{T}{2} + \dfrac{1}{c + v}$

Find $(T_2 - T_1)$]

Suggested Further Reading

Sherwin C. W.: *Concept of Physics* (Holt, New York 1961)
R.S. Shankland, '*Michelson Morley Experiment*', Am. J. Physics **32**, 16 (1964).
E. Whitaker: *History of the Theories of Aether and Electricity* (2 Vols.) (Harper & Row New York, 1960).
A. Stewart, "*The Discovery of Stellar Aberration*", Scientific American **210**(3), 100 (1964).
J.G. Fox, "*Evidence Against Emission Theories*", Am. J. Physics **33**, 1 (1965).

Chapter 3 Lorentz Transformations and Some Consequences

Lorentz transformations which replace G.T. can be derived from the Principle of Relativity together with the Principle of Constancy of the Speed of Light. It then follows immediately that our common sense concepts of space, time etc. have to be radically changed.

3.1 LORENTZ TRANSFORMATIONS (L.T.)

L.T. can be derived on the basis of two postulates or Principles viz. (1) Laws of Physics are of the same form in all inertial frames (Principle of Relativity) and (2) The speed of light in free space has the same value c in all inertial frames (Principle of Constancy of Speed of Light).

(The Principle of Relativity is sometimes stated as: It is impossible by any physical measurements to trace an essential distinction between any two inertial frames which are in relative motion.)

Consider two inertial frames S and S' as shown in Fig. 3.1. S' moves relative to S at a constant speed v along the X-axis. X'-axis is parallel to X-axis and in fact coincides with it.

Suppose times t in S and t' in S' are both measured from the moment when O and O' coincide.

By the Principle of Relativity, linear motion in S (motion under no forces) must appear as linear motion in S'. Therefore, we assume that the transformations which give (x', y', z') in terms of (x, y, z) must also be linear. To an observer in S' a fixed point at x in S appears to move with a velocity v along the negative X-axis. Hence, we may write

$$x' = k(x - vt) \quad \text{... (1a)}$$
$$y' = y \quad \text{... (1b)}$$
$$z' = z \quad \text{... (1c)}$$

where k is a scale factor to be determined. The last two equations above follow from a complete symmetry in the plane perpendicular to the direction of motion.

According to the Principle of Relativity, the relationship between x and x' must be symmetric.

$$\therefore \quad x = k(x' + vt') \quad \text{... (1d)}$$

where the sign of v has been changed since the frames S' is moving with speed v along the positive X-axis relative to S.

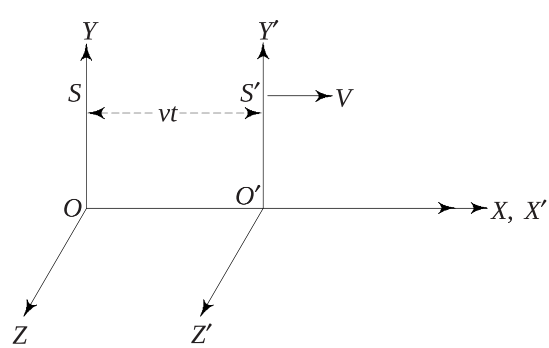

Fig. 3.1 Lorentz Transformations

Let us suppose that at the instant $t = t' = 0$, a spherical light wave starts from the common origin of S and S'. Its progress along the X-axis is described in S and S' respectively by

$$x = ct \quad \text{... (2a)}$$

and

$$x' = ct' \quad \text{... (2b)}$$

since the speed of light is c in both the frames.

$$\therefore \quad ct = x = k(x' + vt') \text{ using (1d)}$$
$$= k(ct' + vt') \text{ using (2d)}$$
$$= k(c + v)t' \quad \text{... (3a)}$$

Similarly

$$ct' = x' = k(x - vt) \text{ using (1a)}$$
$$= k(ct - vt) \text{ using (2a)}$$
$$= k(c - v)t \quad \text{... (3b)}$$

Multiplying Eqns. (3a) and (3b),

$$c^2tt' = k^2(c^2 - v^2)tt'$$

$$\therefore \quad k^2 = \frac{c^2}{c^2 - v^2} = \left(\frac{c^2 - v^2}{c^2}\right)^{-1} = \left(1 - \frac{v^2}{c^2}\right)^{-1}$$

$$\therefore \quad k = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = (1 - \beta^2)^{-1/2} = \Gamma \quad \text{... (4a)}$$

where

$$\beta = \frac{v}{c}$$

From Eqn. (1d),

$$t' = \frac{x - kx'}{kv} = \frac{x - k^2(x - vt)}{kv} \text{ using (1a)}$$

$$= \frac{x - k^2x + k^2vt}{kv} = \frac{k(x - k^2x + k^2vt)}{k^2v} = k\left[t - \frac{x(k^2 - 1)}{k^2v}\right]$$

$$= k\left(t - \frac{vx}{c^2}\right) \quad \text{... (5)}$$

because

$$\frac{k^2 - 1}{k^2} = \beta^2 = \frac{v^2}{c^2}$$

Substituting value of k = Γ, the set of Eqns. (1a, 1b, 1c and 5) may be re-written as

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \Gamma(x - vt) \quad \text{... (6a)}$$

$$y' = y \quad \text{... (6b)}$$

$$z' = z \quad \text{... (6c)}$$

and $$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \Gamma\left(t - \frac{vx}{c^2}\right) \qquad \text{... (6d)}$$

These equations are known as Lorentz Transformations. The inverse equations may be readily written down by interchanging primed and unprimed quantities and replacing v by $-v$. They are

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = \Gamma(x' + v't) \qquad \text{... (7a)}$$

$$y = y' \qquad \text{... (7b)}$$

$$z = z' \qquad \text{... (7c)}$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \Gamma\left(t' + \frac{vx'}{c^2}\right) \qquad \text{... (7d)}$$

Note that: (i) There is an *appreciable* quantitative difference between L.T. and G.T. *only when v/c is appreciable*, that is only at high speeds. (ii) In the limiting case $c = \infty$, L.T. reduce to G.T. and in particular $t = t'$. (iii) Eqns. (6a) and (6d) or (7a) and (7b) show that space and time get 'mixed up' in going from one inertial frame to another. Distinction between space and time is apparently less rigid or a little blurred in relativity! We shall have more to say about this in a later chapter. (iv) If $v > c$, then Γ is imaginary and we face the problem of imaginary spaces and imaginary times. It is impossible therefore to think of two inertial frames relative to each other (in real space and time) at a speed $v > c$!

3.2 SPACE CONTRACTION (LORENTZ CONTRACTION)

Suppose a rod is at rest on the XX' axis in the frame S'. Frame S' (and so also the rod) is moving with a velocity of v along the positive X-axis with respect to frame S. See Fig. 3.2.

Let the two ends of the rod be labelled x_2' and x_1' in frame S' and x_2 and x_1 in the frame S. Then $(x_2' - x_1') = L_0$ say is the length of the rod in the frame S' in which it is at rest.

By L.T., we have $$L_0 = x_2' - x_1' = \frac{(x_2 - vt_2) - (x_1 - vt_1)}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

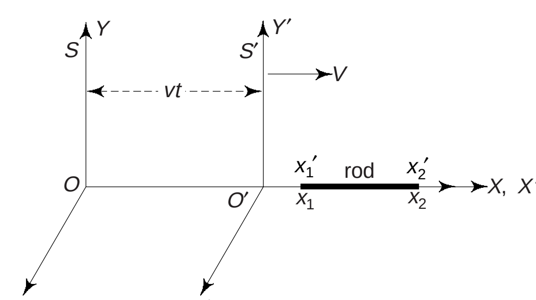

Fig. 3.2 Space Contraction

where t_2 and t_1 are the times at which the observer in S measures the end points. Now to measure the length $L = (x_2 - x_1)$ in S frame, it is necessary that the end point coordinates x_1 and x_2 are measured simultaneously in S. (Remember, length is meaningless otherwise since the rod is in motion for observer in frame S). That is $t_1 = t_2$.

The above equation now becomes

$$L_0 = \frac{x_2 - x_1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{or} \quad L = L_0\sqrt{1-\frac{v^2}{c^2}} \qquad \text{... (1)}$$

The observer is S finds the moving rod to be shorter by the factor $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ as compared to L_0 measured in S' where the rod is at rest. This shortening of length for moving objects is called *length contraction* or *space contraction* or *Lorentz Contraction.*

We see that *length of an object is not absolute* but depends upon the relative velocity of the object and the observer. *Proper Length* of an object is defined as its length measured in a reference frame in which the object is at rest. All observers who are moving with respect to the object (or with respect to whom the object is moving) will find that its length is shorter than its proper length.

3.3 TIME DILATION

Consider two inertial frames S and S' shown in Fig. 3.1 S' moves relative to S with speed v along the positive X direction.

We define an event as an occurrence which takes place at some instant t at a point (x, y, z), for example arrival of a particle at time t at a point (x, y, z) or an electric bulb at (x, y, z) flashing at time t etc.

Suppose a clock at rest at (x', o, o) in frame S' sends one light flash at time t_1' and the next one at a later time t_2'. In S', the time between the two events (consecutive flashes) is $T_0 = t_2' - t_1'$.

The corresponding times as observed by an obsesrver in S on his clock (at rest in S) are respectively:

$$t_1 = \Gamma\left(t_1' + \frac{vx'}{c^2}\right)$$

$$t_2 = \Gamma\left(t_2' + \frac{vx'}{c^2}\right) \text{ as per L.T.}$$

Hence the time interval between the two events (two consecutive flashes) as measured in S is

$$T = t_2 - t_1 = \Gamma\left(t_2' - t_1'\right) = \Gamma T_0 \qquad \text{... (1)}$$

That is

$$T = \frac{T_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \qquad \text{... (2)}$$

In particular if T_o is the unit of time in S', then

$$T = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} > 1$$

Time interval as recorded by an observer in S between two 'moving events' in S is *longer* than the time interval recorded by the observer in S' on a clock which is at rest with respect to where the events occur. The smallest value for the time interval between two events is measured in the frame where the two events occur at the same location. All observers moving with respect to this frame will measure the time interval to be longer. This relativistic effect is called *time dilation*. For example, if the observer in S' in the above case finds that the two flashes occurred 1 second apart, and if $\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 2$, then the observer in S would find that the flashes occurred 2 seconds apart. The time intervals are *not absolute*.

A moving clock always appears to run more slowly than an identical clock at rest with respect to the observer. The time interval dilates, that is expands for a clock in motion. This effect is called *time dilation*.

3.4 RECIPROCITY BETWEEN OBSERVERS

Just as a rod at rest in S' appears to be contracted to an observer in S, so also it can be shown that a rod at rest is frame S appears to be contracted by the same amount as measured by the observer in S'.

Similarly, to an observer in frame S, clocks at rest in S' appear to go slow and to an observer in S', clocks at rest in S appear to go slow. There is thus a complete reciprocity between the observers in frames S and S' as regards contraction in length and dilation of time.

3.5 COMPOSITION OF VELOCITIES

Consider an object moving with a constant velocity U' in an inertial frame S'. The object has velocity components U_x', U_y' and U_z' in S'.

Let us find the velocity of the object as observed by an observer in frame S if S' is moving with a constant speed v along X-axis relative to frames S as shown in Fig. 3.3.

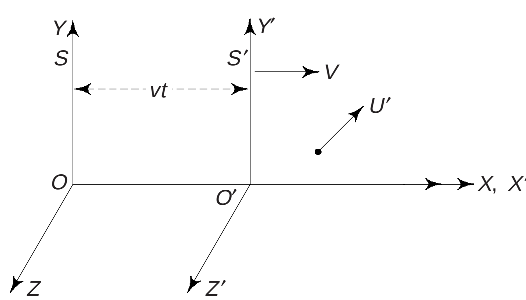

Fig. 3.3 Composition of Velocities

The coordinates and time in the two frames S and S' are related by L.T. In particular,

$$x = \Gamma(x' + vt');\ y = y';\ z = z' \text{ and } t = \Gamma\left(t' + \frac{vx'}{c^2}\right)$$

where $$\Gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

Taking differentials,

$$dx = \Gamma(dx' + vdt');\ dy = dy';\ dz = dz' \quad \text{and} \quad dt = \Gamma\left(dt' + \frac{vdx'}{c^2}\right)$$

In frame S, the velocity components of the object are

$$U_x = \frac{dx}{dt}\ ;\ U_y = \frac{dy}{dt} \quad \text{and} \quad U_z = \frac{dz}{dt}$$

From above equations we get

$$U_x = \frac{dx}{dt} = \frac{\Gamma(dx' + vdt')}{\Gamma\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{dx' + vdt'}{dt'\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)}$$

$$= \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{U_x' + v}{1 + \frac{v}{c^2}U_x'} \qquad \text{... (1a)}$$

Similarly,
$$U_y = \frac{dy}{dt} = \frac{U_y'}{\Gamma\left(1 + \frac{v}{c^2}U_x'\right)} \qquad \text{... (1b)}$$

And
$$U_z = \frac{dz}{dt} = \frac{U_z'}{\Gamma\left(1 + \frac{v}{c^2}U_x'\right)} \qquad \text{... (1c)}$$

Note that Eqns. (1a), (1b) and (1c) are quite different from Galilean case viz.

$$U_x = U_x' + v,\ U_y = U_y' \text{ and } U_z = U_z'$$

Inverse transformations for the velocity components are obtained by changing sign of v. They are

$$U_x' = \frac{U_x - v}{1 - \frac{v}{c^2}U_x} \qquad \text{... (2a)}$$

$$U_y' = \frac{U_y}{\Gamma\left(1 - \frac{v}{c^2}U_x\right)} \qquad \text{... (2b)}$$

$$U_z' = \frac{U_z}{\Gamma\left(1 - \frac{v}{c^2}U_x\right)} \qquad \text{... (2c)}$$

Some interesting conclusions are:

(A) The Speed of an Object can Never Exceed the Speed of Light *c*.

To see this let $U_x' = kc$ where k is a positives constant. Let us suppose that S' travels at a speed $v = kc$ relative to S. Then according to Galilean theory, the velocity $U_x = 2kc$. However, according to Eqn. (1a)

$$U_x = \frac{U_x' + v}{1 + \frac{v}{c^2}U_x'} = \frac{kc + kc}{1 + \frac{kc}{c^2}\cdot kc} \quad \because \quad U_x' = v = kc.$$

$$= c\left(\frac{2k}{1+k^2}\right)$$

The factor $\frac{2k}{1+k^2}$ has a maximum value of unity when $k = 1$. (Convince yourself of this by trying different positive values for k). Thus U_x can never exceed c.

(B) The Speed of Light *c* is the Same in all Directions in all Inertial Frames

Consider a light signal travelling with speed c along the X'-axis in frame S'. According to Galilean theory the speed of light as observed in S would be $(c + v)$. However according to Eqn. (1a)

$$U_x = \frac{U_x' + v}{1 + \frac{v}{c^2}U_x'} = \frac{c+v}{1 + \frac{v}{c^2}c}$$

$$= \frac{c+v}{1+\frac{v}{c}} = \frac{c+v}{\frac{1}{c}(c+v)} = c$$

Consider next the case of observers moving at right angles to the direction of light propagation.

Suppose a ray of light travels along $O'Y'$ in frames S'. Then $U_x' = 0$, $U_y' = c$, $U_z' = 0$

Now $$U_x = \frac{U_x' + v}{1 + \frac{v}{c^2}U_x'} = v \quad \because \quad U_x' = 0$$

$$U_y = \frac{U_y'}{\Gamma\left(1 + \frac{v}{c^2}U_x'\right)} = \frac{c}{\Gamma} \quad \because \quad U_y' = c,\ U_x' = 0$$

$$U_z = \frac{U_z'}{\Gamma\left(1 + \frac{v}{c^2}U_x'\right)} = 0 \quad \because \quad U_z' = 0$$

$\therefore$ $$U = \sqrt{U_x^2 + U_y^2 + U_z^2} = \sqrt{U_x^2 + U_y^2} = \left(v^2 + \frac{c^2}{\Gamma^2}\right)^{1/2}$$

$$= \left[v^2 + c^2\left(1 - \frac{v^2}{c^2}\right)\right]^{1/2} = [v^2 + c^2 - v^2]^{1/2} = c$$

(C) *c* plus *c* = *c*!

If we substitute $U_x' = c$ and $v = c$ in Eqn. (1a), we get

$$U_x = \frac{U_x' + v}{1 + \frac{v}{c^2}U_x'} = \frac{c + c}{1 + \frac{c}{c^2}c} = c.$$

The student should convince himself that relative velocity of two objects or two frames or an object and a frame cannot exceed c.

SUMMARY

L.T. which replace G.T. are derived on the basis of two assumptions viz. (i) Laws of Physics are of the same form in all inertial frames (ii) The speed of light in free space has the same value c in all inertial frames.

L.T. are:

$$x' = \Gamma(x - vt) \qquad x = \Gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \Gamma\left(t - \frac{vx}{c^2}\right) \qquad t = \Gamma\left(t' + \frac{vx'}{c^2}\right)$$

Length of an object is not absolute but depends upon the relative velocity of the object and the observer. Proper length (L_0) of an object is its length measured in a reference frame in which the object is at rest. All inertial observers who are in motion relative to the object find its length to be shorter than its proper length L_0.

$$L = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2} < L_0$$

Time intervals are not absolute. A moving clock runs more slowly than an identical clock at rest with respect to the observer. The time interval dilates, that is, it expands for a clock in motion. This relativistic effect is called time dilation.

$$T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} > T_0$$

There is a reciprocity between observers in S and S' as regards length contraction and time dilation. The law of composition of velocities is different from its Galilean counterpart.

$$U_x = \frac{U_x' + v}{1 + \frac{v}{c^2}U_x'} \qquad U_x' = \frac{U_x - v}{1 - \frac{v}{c^2}U_x}$$

$$U_y = \frac{U_y'}{\Gamma\left(1 + \frac{v}{c^2}U_x'\right)} \qquad U_y' = \frac{U_y}{\Gamma\left(1 - \frac{v}{c^2}U_x\right)}$$

$$U_z = \frac{U_z'}{\Gamma\left(1+\frac{v}{c^2}U_x'\right)} \qquad U_z' = \frac{U_z}{\Gamma\left(1-\frac{v}{c^2}U_x\right)}$$

As a result of the relativistic laws of composition of velocities, it follows that the speed of an object cannot exceed c.

ILLUSTRATIVE EXAMPLES

Example 1 Calculate $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ when (a) $v = 10^{-2}c$ and (b) $v = 0.9998c$.

Solution According to the binomial expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \approx 1 + nx \text{ when } x \ll 1$$

(a) Setting $x = -\frac{v^2}{c^2} = 10^{-4}$ and $n = \frac{1}{2}$, we have

$$\left(1-\frac{v^2}{c^2}\right)^{1/2} = (1 - 10^{-4})^{1/2} \approx 1 + \frac{1}{2}(-10^{-4}) = 1 - 0.00005 = 0.99995$$

(b) $\left(1-\frac{v^2}{c^2}\right)^{1/2} = (1 - (0.9998)^2)^{1/2} = [1 - (1 - 0.0002)^2]^{1/2}$

Now $\qquad (1 - 0.0002)^2 \approx 1 + 2(-0.0002) = 1 - 0.0004$

$$\therefore \qquad \left(1-\frac{v^2}{c^2}\right)^{1/2} = [1 - (1 - 0.0004)]^{1/2} = (0.0004)^{1/2} = 0.02.$$

Example 2 What is the velocity of a metre scale if its length is observed to be 0.99 m?

Solution $\qquad L = L_0\left(1-\frac{v^2}{c^2}\right)^{1/2}$ or $\left(1-\frac{v^2}{c^2}\right) = \left(\frac{L}{L_0}\right)^2 = (0.99)^2$

$$\therefore \qquad \frac{v^2}{c^2} = 1 - (0.99)^2 = 1 - (1 - 0.01)^2 \approx 1 - [1 - 2\,(0.01)] = 0.02$$

$$\therefore \qquad v = 0.141c$$

Example 3 How long does it take for a metre scale to pass you if it is travelling with a speed of $0.6c$ relative to you along the direction of its length?

Solution The time required is $t = \frac{\text{length}}{\text{velocity}}$, where length L is the contracted length $L = L_0\left(1-\frac{v^2}{c^2}\right)^{1/2}$

$$\therefore \qquad t = \frac{L_0\left(1-\dfrac{v^2}{c^2}\right)^{1/2}}{v} = \frac{1[1-(0.6)^2]^{1/2}}{0.6\times 3\times 10^8} = \frac{0.8}{1.8\times 10^8} = 4.44 \times 10^{-9} \text{ s.}$$

Example 4 An astronaut is travelling in a space vehicle with velocity 0.6c relative to the earth. The astronaut measures his pulse rate to be 75 per minute. Signals generated by astronaut's pulse are radioed to earth when the space vehicle is moving perpendicular to a line that connects the vehicle with an earth observer. What is the pulse rate as measured by the observer on the earth? Hence comment on the life span of the astronaut from point of view of the earth observer.

Solution The time interval between two consecutive pulses as measured by the astronaut is $T_0 = \dfrac{60}{75} = 0.8$ s. The interval between the pulses as measured by the earth observer is

$$T = \frac{T_0}{\left(1-\dfrac{v^2}{c^2}\right)^{1/2}} = \frac{0.80}{(0.64)^{1/2}} = 1.0 \text{ s}$$

The earth observer measures a pulse rate of 60/1.0 = 60 pulses per minute. The pulse rate measured by the earth observer is less than that measured by the astronaut. Hence the life span of the astronaut determined by the total number of his heartbeats is longer as measured by the earth clock than the life span measured by a clock aboard the space vehicle.

Note that beating of the heart is a kind of clock mechanism. The repetitive radio signal from the space vehicle is subject to time dilation as well as Doppler effect. We have eliminated the usual Doppler effect by choosing to calculate the pulse rate at the instant when the vehicle is travelling perpendicular (transverse) to the line connecting the vehicle and the earth observer. The time dilation still brings about a shift in the frequency. Hence time dilation brings about a transverse Doppler shift. We shall study Doppler effect in detail later.

Example 5 Length of a moving rod can be defined as the product of its velocity and the time interval between the instant that one end point of the rod passes a fixed mark and the instant that the other end point passes the same mark. Show that this definition leads to space contraction.

Solution Suppose the rod is at rest in frame S' and its end point coordinates are x_1' and x_2'.

We are in frame S at the fixed mark x say and watch the rod go by us with speed v. Then by L.T.

$x_1' = \Gamma(x_1 - vt_1)$ and $x_2' = \Gamma(x_2 - vt_2)$ where x_1 and x_2 in frame S are measured at time t_1 and t_2 respectively. Since the mark is fixed, $x_1 = x_2 = x$.

$$\therefore \qquad \frac{x_1'}{\Gamma} + vt_1 = x_1 = x_2 = \frac{x_2'}{\Gamma} + vt_2$$

$$\therefore \qquad v(t_2 - t_1) = \frac{x_1' - x_2'}{\Gamma}$$

$(x_1' - x_2') = L_0$, the length measured in frame S' in which the rod is at rest. $v(t_2 - t_1)$ is the length L measured by us in frame S as per definition.

$$\therefore \qquad L = \frac{L_0}{\Gamma} = L_0\left(1-\frac{v^2}{c^2}\right)^{1/2} \text{ as required.}$$

Example 6 Charged pions are unstable and when brought to rest their half-life is measured to be 1.8×10^{-8} s, that is, half the number present at any time will have decayed 1.8×10^{-8} s later. A beam of pions travelling at a speed of 0.99 C is found to drop to half its original intensity in travelling a distance of 37.5 metres. If we take the pion speed to be $0.99c = 2.970 \times 10^8$ m/s, then the distance travelled in the course of half-life 1.8×10^{-8} s should be $(2.970 \times 10^8)(1.8 \times 10^{-8}) = 5.34$ m. This seems to contradict the experimentally measured distance of 37.5 m. Why? (a) show how length contraction accounts for the apparent contradiction. (b) Show how time dilation accounts for the contradiction.

Solution Contradiction arises because velocity (2.970×10^8 m/s) and distance (37.5 m) are measured in the laboratory frame whereas the time used (1.8×10^{-8} s) is measured in the rest frame of pions, that is frame in which pions are at rest.

(a) For an observer in a frame in which the pions are at rest, the laboratory is travelling at a speed of 0.99c relative to him. To him the distance $L_0 = 37.5$ m as measured in the laboratory appears to be contracted to,

$$L = L_0 \left(1 - \frac{v^2}{c^2}\right)^{1/2} = 37.50\ [1 - (0.99)^2] = 5.346 \text{ m.}$$

The time elapsed in covering this distance is

$$T_0 = \frac{5.346}{2.97 \times 10^8} = 1.8 \times 10^{-8} \text{ s}$$

Thus the discrepancy is removed when all quantities are measured in the frame of the observer who is at rest with respect to the pions.

(b) In the laboratory frame, a time interval of $T_0 = 1.8 \times 10^{-8}$ s which is measured in pion rest frame is dilated to

$$T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = \frac{1.8 \times 10^{-8}}{[1 - (0.99)^2]^{1/2}} = 12.62 \times 10^{-8} \text{ s}$$

In the laboratory frame, the half-life of pions is 12.62×10^{-8} s, they appear to live longer.

Distance travelled by pions travelling at 2.97×10^8 m/s in the above dilated time is (2.97×10^8) $(12.62) = 37.50$ m.

Once again we find the discrepancy is removed when all quantities are measured in the laboratory frame.

Example 7 On their 18th birthday, one twin Seeta jumps on a moving sidewalk, which takes her to a distant station 4 light years away with a speed of 0.8c. On reaching the station she immediately jumps onto the returning moving sidewalk and comes back to earth again at the speed of 0.8c. On her return to earth compare the age of Seeta with that of her sister Geeta who stayed on earth.

Solution Light year is the distance travelled by light in one year. It is equal to $(365 \times 24 \times 60 \times 60 \times 3 \times 10^8)$ m.

According to Geeta (on earth) the time required for the trip to the station is

$$T = \frac{\text{Distance}}{\text{Speed}} = \frac{4(365 \times 24 \times 60 \times 60 \times 3 \times 10^8)}{0.8 \times 3 \times 10^8}$$

$$= 5(365 \times 24 \times 60 \times 60) \text{ s} = 5 \text{ years.}$$

Since the return trip is made with the same speed, the time for return trip is also 5 years. In other words, the total time for the round trip according to Geeta is 2T that is 10 years. Therefore Geeta is 28 years old when her twin sister Seeta arrives back to earth.

According to Seeta the distance between the earth and the station is not 4 light years but is contracted to $L = L_0 \left(1 - \frac{v^2}{c^2}\right)^{1/2} = 4[1 - (0.8)^2] = 2.4$ light years.

Journey to the star takes time

$$T' = \frac{2.4(365 \times 24 \times 60 \times 60 \times 3 \times 10^8)}{0.8 \times 3 \times 10^8} = 3 \text{ years}$$

Hence the total time for the round trip is only 6 years. In other words, Seeta is 24 years old on her arrival back. Seeta is thus younger than her twin sister by 4 years.

Alternatively it may be seen that the time measured by Seeta is less than the time measured by Geeta on account of time dilation. Time for the trip to the station according to Seeta is

$T_0 = T\left(1 - \frac{v^2}{c^2}\right)^{1/2} = 5[1 - (0.8)^2]^{1/2} = 3$ years and the total time of travel is 6 years as compared to 10 years for Geeta.

Note that the motion of twins is not symmetrical. It cannot be argued that because of reciprocity Geeta should be younger than Seeta by 4 years. Whereas Geeta is in an inertial frame throughout the trip, the same is not true of Seeta. During initial acceleration to speed 0.8 c, reversal of speed at the station and the period of deceleration prior to reaching back on earth, Seeta is essentially in a non-inertial frame. Calculations done by taking into account the periods of accelerations and decelerations though beyond the scope of this book, give the same result as obtained above by cutting the problem down to our size!

Example 8 It takes 10^5 years for light to reach us from the farthest part of our galaxy. Is it possible for a man to travel out to that part of our galaxy at a constant speed in a reasonable time of say 50 years?

Solution The distance travelled by light in 10^5 years is according to an observer on earth given by $L_0 = 10^5$ Nc, where c is the speed of light (3×10^8 m/s) and N is the number of seconds in one year.

For the traveller who observes the farthest part of the galaxy approaching him at a speed say v, the distance is

$$L = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2} = 10^5 \text{ Nc}\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

Total time available for the journey is 50 N seconds. Hence the speed necessary is given by

$$v = \frac{L}{50\text{ N}} = \frac{10^5 \text{ Nc}\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{50\text{ N}}$$

$$\therefore \quad \frac{v}{c} = 2 \times 10^3 \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$\therefore \quad \frac{v^2}{c^2} = 4 \times 10^6 \left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore \qquad \frac{v^2}{c^2} = \frac{4\times 10^6}{1+4\times 10^6} \quad \text{or} \quad \frac{v}{c} = \left(\frac{1}{1+2.5\times 10^{-7}}\right)^{1/2}$$

$$= (1 + 2.5 \times 10^{-7})^{-1/2} \approx 1 - 1.25 \times 10^{-7} \text{ using binomial theorem.}$$

$$\therefore \qquad v = 0.99999975c.$$

A traveller travelling at above speed will be able to complete the trip in 50 years. The traveller ages by 50 years in the course of the journey. The time for the journey as measured on the earth will be more than 10^5 years!

Example 9 Two light pulses are moving in the positive direction along the *X*-axis of the frame *S*, the distance between them being *d*. What is their distance of separation as seen from frame S' moving with speed *v* along positive *X*-axis with respect to frame *S*?

Solution Suppose one pulse passes the origin *O* at time $t = 0$ in *S*. Its equation of motion is

$$x = ct \qquad \text{... (1)}$$

The equation of the other pulse may be written as

$$x = ct + d \qquad \text{... (2)}$$

In frame S', Eqn. (1) becomes

$$\Gamma(x' + vt') = c\Gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$\therefore \qquad x'\left(1-\frac{v}{c}\right) = ct'\left(1-\frac{v}{c}\right) \quad \text{or} \quad x' = ct' \qquad \text{... (3)}$$

Similarly Eqn. (2) becomes

$$\Gamma(x' + vt') = c\Gamma\left(t' + \frac{vx'}{c^2}\right) + d$$

$$\text{or} \qquad x' = ct' + \frac{d}{\Gamma\left(1-\frac{v}{c}\right)} \quad \text{or} \quad x' = ct' + d' \text{ say} \qquad \text{... (4)}$$

Eqns. (3) and (4) show that in S' the pulses travel with speed *c* (as they should) and the distance between them is

$$d' = \frac{d}{\Gamma\left(1-\frac{v}{c}\right)} = \frac{d\left(1-\frac{v^2}{c^2}\right)^{1/2}}{\left(1-\frac{v}{c}\right)} = d\left(\frac{c+v}{c-v}\right)^{1/2}$$

Example 10 A rigid rod of length *L* makes an angle θ with the *X*-axis of the system in which it is at rest in the $X - Y$ plane. Show that for an observer moving with respect to the rod with speed *v* along the positive *X*-direction; the apparent length L' and the angle θ' are given by

$$L' = L\left[\left(\frac{\cos\theta}{\Gamma}\right)^2 + \sin^2\theta\right]^{1/2}; \quad \tan\theta' = \Gamma\tan\theta$$

Solution Suppose the rod is at rest in frame S as shown in Fig. 3.4. Coordinates of the ends of the rod in frame S are:

$$(x_A,\ y_A = 0) \quad \text{and} \quad (x_B = L\cos\theta + x_A,\ y_B = L\sin\theta)$$

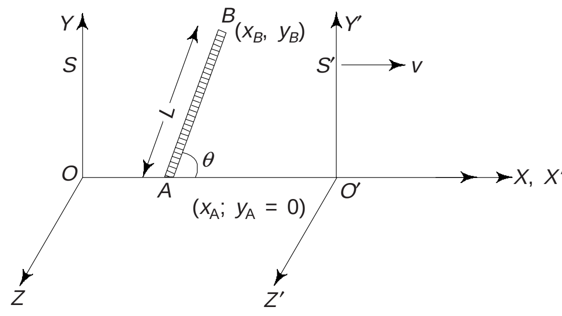

Fig. 3.4 For Illustrative Example 10

Let (x'_A, y'_A) and (x'_B, y'_B) denote the corresponding coordinates at some instant t' as observed in S'. From L.T., we have

$$x_A = \Gamma(x'_A + vt'),\ y_A = 0 = y'_A$$

$$x_B = L\cos\theta + x_A = \Gamma(x'_B + vt'),\ y_B = L\sin\theta = y'_B$$

$$\therefore \qquad x'_A = \frac{x_A}{\Gamma} - vt' \quad \text{and} \quad x'_B = \frac{L\cos\theta + x_A}{\Gamma} - vt'$$

$$\therefore \qquad (x'_B - x'_A) = \frac{L\cos\theta}{\Gamma};\ y'_B - y'_A = L\sin\theta$$

Length of the rod in frame S' is

$$L' = \left[(x'_B - x'_A)^2 + (y'_B - y'_A)^2\right]^{1/2} = L\left[\left(\frac{\cos\theta}{\Gamma}\right)^2 + \sin^2\theta\right]^{1/2}$$

Angle θ' as measured in S' is given by

$$\tan\theta' = \frac{y'_B - y'_A}{x'_B - x'_A} = \frac{L\sin\theta}{L\dfrac{\cos\theta}{\Gamma}} = \Gamma\tan\theta$$

Note that the rod is both contracted in length and rotated in space as observed in frame S'.

Example 11 Two rockets A and B are travelling to the right and left with velocities of 0.8c and 0.6c respectively as observed by an observer on earth. (a) What is their relative velocity as judged by the observer on earth? (b) What is the velocity of rocket A relative to rocket B?

Solution

(a) According to the observer on earth, the relative velocity of the two rockets moving in opposite directions is 0.8c + 0.6c = 1.4c. Note that this is *NOT* the velocity of one inertial frame as observed by an observer in another inertial frame.

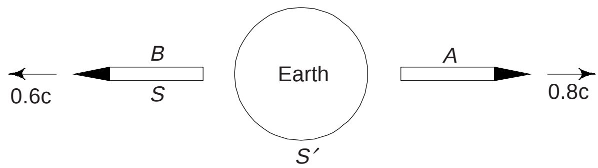

Fig. 3.5 For Illustrative Example 11

(b) Let S and S' be the frames of rocket B and the earth respectively as shown in Fig. 3.5. In frame S', the rocket A is moving with a speed $U_x' = 0.8$ c.

From frame S, the earth frame S' is observed to travel to the right with a speed of $v = 0.6$ c. The speed of rocket A as observed in frame S is

$$U_x = \frac{U_x' + v}{1 + \frac{vU_x'}{c^2}} = \frac{(0.8 + 0.6)c}{1 + \frac{(0.8)(0.6)c^2}{c^2}} = 0.94 \text{ c}$$

Example 12 An observer on a railway platform sees two trains approaching each other at a speed of 7/5 c. An observer on one train sees the other train approaching him with a speed of (35/37) c. What are the velocities of the trains relative to the observer on the platform?

Solution Let U' and v denote the speeds of trains A and B relative to the platform as shown in Fig. 3.6.

Fig. 3.6 For Illustrative Example 12

Let the frames of the train B and the platform be denoted by S and S' respectively. Then $U' + v = 7/5$ c.

In frame S', the train A is observed to be travelling towards the right with a speed of $U_x' = U'$. Its speed as observed in frame S moving to the left of the platform (frame S') is

$$U_x = \frac{U_x' + v}{1 + \frac{vU_x'}{c^2}} = \frac{U' + v}{1 + \frac{vU'}{c^2}} = \left(\frac{35}{37}\right)c$$

$$\therefore \quad \frac{7/5\,c}{1 + \frac{vU'}{c^2}} = \left(\frac{35}{37}\right)c \quad \text{or} \quad \frac{1}{1 + \frac{vU'}{c^2}} = \left(\frac{25}{37}\right)$$

$$\therefore \quad \frac{vU'}{c^2} = \frac{12}{25}$$

Also $$\left(\frac{U'}{c} + \frac{v}{c}\right)^2 = \frac{49}{25} \text{ because } U' + v = 7/5 \text{ c}$$

$\therefore \qquad \left(\frac{U'}{c} - \frac{v}{c}\right)^2 = \left(\frac{U'}{c} + \frac{v}{c}\right)^2 - \frac{4U'v}{c^2} = \frac{49}{25} - \frac{48}{25} = \frac{1}{25}$

$\therefore \qquad \frac{U'}{c} - \frac{v}{c} = \frac{1}{5}$

Since $\qquad \frac{U'}{c} + \frac{v}{c} = \frac{7}{5}$

on solving the two simultaneous equations we get $U' = \frac{4}{5}$ c and $v = \frac{3}{5}$ c.

Example 13 Show that the velocities U' and U measured in frames S' and S are related by

$$\sqrt{1 - \frac{U'^2}{c^2}} = \sqrt{\frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{U^2}{c^2}\right)}{\left(1 - \frac{vUx}{c^2}\right)^2}}$$

where v is the velocity of S' relative to S. What is the inverse relation?

Solution $\qquad U'^2 = U_x'^2 + U_y'^2 + U_z'^2$

$$= \frac{(U_x - v)^2 + \left(U_y^2 + U_z^2\right)\Gamma^{-2}}{\left(1 - \frac{vU_x}{c^2}\right)^2} \text{ from Eqns. } (2a, b, c) \text{ of Art. 3.5.}$$

$$= \frac{(U_x - v)^2 + \left(U_y^2 + U_z^2\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{vU_x}{c^2}\right)^2}$$

$$\therefore \qquad 1 - \frac{U'^2}{c^2} = 1 - \frac{\left(\frac{U_x}{c} - \frac{v}{c}\right)^2 + \left(\frac{U^2}{c^2} - \frac{U_x^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{vU_x}{c^2}\right)^2}$$

$$= \frac{1 - \frac{2vU_x}{c^2} + \frac{v^2U_x^2}{c^4} - \frac{U_x^2}{c^2} + \frac{2vU_x}{c^2} - \frac{v^2}{c^2} - \frac{U^2}{c^2} + \frac{U_x^2}{c^2} + \frac{U^2v^2}{c^4} - \frac{v^2U_x^2}{c^4}}{\left(1 - \frac{vU_x}{c^2}\right)^2}$$

$$= \frac{1 - \dfrac{v^2}{c^2} - \dfrac{U^2}{c^2} + \dfrac{U^2 v^2}{c^4}}{\left(1 - \dfrac{vU_x}{c^2}\right)^2} = \frac{\left(1 - \dfrac{v^2}{c^2}\right)\left(1 - \dfrac{U^2}{c^2}\right)}{\left(1 - \dfrac{vU_x}{c^2}\right)^2}$$

$$\therefore \qquad \sqrt{1 - \frac{U'^2}{c^2}} = \sqrt{\frac{\left(1 - \dfrac{v^2}{c^2}\right)\left(1 - \dfrac{U^2}{c^2}\right)}{\left(1 - \dfrac{vU_x}{c^2}\right)^2}}$$

The inverse relation can be written down by changing the sign of v. It is

$$\sqrt{1 - \frac{U^2}{c^2}} = \sqrt{\frac{\left(1 - \dfrac{v^2}{c^2}\right)\left(1 - \dfrac{U'^2}{c^2}\right)}{\left(1 + \dfrac{vU'_x}{c^2}\right)^2}}$$

EXERCISES

1. State the basic postulates of relativity and derive the L.T.
2. What do you understand by space contraction? Derive the formula for it.
3. What is time dilation? How does it arise from transformations of time and space coordinates recorded by two inertial observers in relative motion?
4. What do you understand by reciprocity between two inertial frames in relative motion as regards contraction of space and dilation of time?
5. Write down L.T. hence obtain the relativistic law of composition of velocities.
6. Write down the relativistic law of addition of velocities and explain what important conclusions follow therefrom.
7. Show that two successive L.T. corresponding to speeds v_1 and v_2 are equivalent to a single L.T. corresponding to a speed of $v = \dfrac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$.

   Hint: Consider three inertial frames S, S', S''. Let S' move with speed v_1 relative to S and let S'' move with speed v_2 relative to S'. Then

   $$x' = \Gamma_1 (x - v_1 t) \text{ etc. and } x'' = \Gamma_2 (x' - v_2 t') \text{ etc.}$$

   Show that
   $$x'' = \Gamma_1 \Gamma_2 \left[x\left(1 + \frac{v_1 v_2}{c^2}\right) - (v_1 + v_2)t \right]$$
   $$= \Gamma_1 \Gamma_2 \left(1 + \frac{v_1 v_2}{c^2}\right)\left[x - \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}} t \right]$$

$$= \Gamma(x - vt) \quad \text{where} \quad \Gamma = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \quad \text{and} \quad v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

8. Show that space contraction can be derived by assuming that an observer in frame S measures the time interval between the passing of his origin by the two ends of a metre stick in frame S'. The length in frame S is obtained by multiplying this time by v.
9. Show that time dilation can be derived by measuring the distance in frame S between two events occurring at the origin of frame S' and dividing this distance by v to get the time interval in frame S.
10. A rocket is moving at such a speed that its length as measured by an observer on the earth is only half of its proper length. How fast is the rocket moving relative to the earth? (*Ans.* 0.866 c)
11. How fast must a space ship travel if a traveller in the space ship ages at only half the rate we are ageing on the earth? (*Ans.* 0.866 c)
12. An air plane is moving with respect to earth with a speed of 600 m/s. As determined by earth clock, how long will it take for the air plane clock to fall behind by a microsecond?
(*Ans.* 5×10^5 s = 5.8 days approx.)

Hint: $$\Delta t_{\text{earth}} = \frac{\Delta t_{\text{Plane}}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} ; \; \Delta t_{\text{earth}} - \Delta t_{\text{plane}} \approx 2 \times 10^{-12} \, \Delta t_{\text{earth}} = 10^{-6} \text{ s}$$

13. A particle moving at 0.8 c in a laboratory is observed to decay after travelling 3 m. What is its life span in its rest frame? (*Ans.* 0.75×10^{-8} s)
14. Two space ships each measuring 100 m in its own rest frame pass by each other travelling in opposite directions. Instruments on space ship A determine that the front end of the space ship B required 5.00×10^{-6} s to traverse the full length of A. (a) What is the relative velocity of the two space ships? (b) A clock in the front end of B reads exactly one O'clock as it passes by the front end of A. What will the clock read as it passes by the rear end of A?
(*Ans.* (a) 2×10^7 m/s (b) $1 + 4.99 \times 10^{-6}$ s)
15. Life time of a particle is 2.2×10^{-6} s in a frame in which it is at rest. Calculate the average distance it will travel in vacuo before decaying if its speed is (a) 0.9 c, (b) 099 c and (c) 0.999 c.
(*Ans.* (a) 1.36 km; (b) 4.6 km; (c) 14.5 km)
16. What is the velocity of a metre scale if its length is observed to be shortened by a centimeter?
(*Ans.* v = 0.141c)
17. An observer O' holds a metre scale at an angle of 30° with respect to the positive X'-axis. O' is moving in the positive $X - X'$ direction with a velocity v relative to observer O. What is the value of v if the metre scale makes an angle of 45° with respect to X-axis of O. What is the length of the metre scale as measured by O? (*Ans.* v = 0.816 c, L = 0.707 m)
18. A particle with a life time of 8×10^{-6} s created at an altitude of 10 km travels directly towards the earth. If it decays as soon as it reaches the earth, what is its velocity relative to the earth?
(*Ans.* 0.972 c)
19. A space ship is observed to cover 90 m in 5×10^{-7} s. What is the distance travelled and time taken as measured by an observer in the space ship? (*Ans.* 72 m, 4×10^{-7}s)

20. In frame S' a particle moves with a speed of 2.4×10^8 m/s at an angle of 30° to the X'-axis. If S' is travelling at a speed of 1.8×10^8 m/s along the positive $X - X'$-axis relative to frame S, find the velocity of the particle (magnitude and direction) as observed by an observer in S.
(*Ans.* $v = 2.823 \times 10^8$ m/s making an angle of 13.9° with X-axis)
21. Show that $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$.
22. A space traveller desires to go to a star 5 light years away. What should be the uniform velocity of his rocket if he is to reach there in one year as measured by clocks at rest on the rocket? Neglect the time of acceleration etc. (*Ans.* v = 0.98 c)
23. A man goes to the Polar Star and immediately returns to the earth on a rocket. Calculate the age difference between him and his twin sister who stayed on earth. The velocity of the rocket is 0.8 c and the Polar Star is 40 light years away from the earth. (*Ans.* 40 *y*)
24. Two particles approach one another. Calculate their relative speed if each has a speed of 0.9 c with respect to the laboratory. (*Ans.* 0.995 c)
25. *A* K^o meson at rest decays into π^+ and π^- particles each having a speed of 0.85 c. If a K^o meson travelling at a speed of 0.9 c decays, what is the maximum speed one of the pions can have? What is the least speed? (*Ans.* 0.9915 c, 0.212 c)
26. An observer in frame S observes a photon travelling in a direction making 60° with the X-axis. A frame S' travels with a speed 0.6 c along the common $X - X'$ axis. What angle does the photon make with the X'-axis as observed by an observer in S'? (*Ans.* 81.8° above the negative X'-axis)

    Hint: U_x = 0.500c, U_y = 0.866c. Find U_x', U_y' and $\tan \theta' = \dfrac{U_y'}{U_x'}$.
27. A space traveller on a rocket notes that he is travelling towards a star 1 light year away from the earth with a speed v. An observer on earth finds that the rocket is travelling with a speed v but says the star is 10 light years away. (a) How long does the trip last according to the space traveller? (b) How long does the trip last according to the observer on the earth? Who is correct?
(*Ans.* (a) 1.0005 *y* (b) 10.05 *y*. Both are correct)
28. Assume that the speed of light is 100 km/h. A lady relating the story of the delivery of her baby says she drove herself at 80 km/h to the hospital which was 100 km away as per the highway board. She also claims that according to her watch the baby was born 1 hour after she left. Show that the baby was born in the hospital.
29. A stationery missile explodes and breaks into equal pieces. The two pieces move with velocities c/2, one to the right and the other to the left. The piece moving to the right again explodes into two pieces such that with respect to its own rest frame, the resulting two pieces take off with velocities c/2 one to the right and the other to the left. Find the velocities of these last two pieces with respect to earth. (*Ans.* 4/5 c and zero)
30. If S' has velocity c relative to S, show that all bodies moving relative to S with speeds less than c have a speed c as observed from frame S'. Consider all motions to be along X-axis only.
31. Two bodies are moving in opposite directions with speeds c relative to an inertial frame. Show that their relative velocity is c.

Suggested Further Reading

W. Rindler, "*Length Contraction Paradox*", Am. J. phys. **29**, 365(1961).
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D.H. Frisch and J.H. Smith, "*Measurement of Relativistic Time Dilation using μ Mesons*", Am. J. Phys. **31**, 342(1963).

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Chapter 4

Relativity of Simultaneity, Colocality, Etc.

INTRODUCTION

It follows from L.T. that the concepts of space and time are not absolute. We are therefore led to revise our concepts of simultaneity, colocality etc.

4.1 EVENTS

An event is defined as an occurrence which takes place at some point (x, y, z) at some instant of time t. Arrival of a particle at a point (x, y, z) at time t, a bulb located at (x, y, z) flashing at time t, a gun at (x, y, z) firing at time t are examples of events. An event is thus described by the set of four numbers (x, y, z, t).

An event has a meaning in an inertial frame but the numbers describing its position and time (x, y, z, t) are different in different inertial frames of reference. Their transformations are given by L.T.

If two events happen at the same place and at the same time, they are called *coincident events.* If two events happen at the same place but not necessarily at the same time, they are called *colocal events.*

If two events happen at the same time but not necessarily at the same place, they are called *simultaneous events.*

4.2 RELATIVITY OF SIMULTANEITY, COLOCALITY, ETC.

Consider two events (x_1', y_1', z_1', t_1') and (x_2', y_2', z_2', t_2') in frame S'.

(A) Coincident Events

If the two events are coincident in S', then $\Delta x' = (x_2' - x_1') = 0$, similarly $\Delta y' = \Delta z' = 0$ (same place). Also $\Delta t' = (t_2' - t_1') = 0$ (same time).

In frame S, the above two events are described by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2). From L.T. we get $\Delta x = (x_2 - x_1) = \Gamma(\Delta x' + v\Delta t') = 0$. Also $\Delta y = \Delta z = 0$.

Similarly $$\Delta t = (t_2 - t_1) = \Gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = 0$$

Thus the events are coincident in frame S also. In other words if two incidents are coincident in one inertial frame, they are coincident in every inertial frame. The statement that two events are coincident is thus true in all inertial frames, that is, it is universally true.

(B) Colocal Events

If the two events are colocal in S', then $\Delta x' = \left(x_2' - x_1'\right) = 0$, $\Delta y' = \Delta z' = 0$ but $\Delta t' = \left(t_2' - t_1'\right) \neq 0$.

In frame S, the two events are described by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2). From L.T. we get

$$\Delta x = (x_2 - x_1) = \Gamma(\Delta x' + v\Delta t') = \frac{v\Delta t'}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = \Gamma v\Delta t' \qquad \text{... (1)}$$

$$\Delta y = \Delta z = 0$$

$$\Delta t = \Gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = \frac{\Delta t'}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = \Gamma\ \Delta t' \qquad \text{... (2)}$$

We see that the two events which happened at the same point in S', do not happen at the same point in frame S . They happen a distance $\Gamma v\Delta t'$ apart. If we disregard the factor Γ for a moment then Eqn. (1) is easy to understand. Consider two successive firings of a fixed gun in a moving train (frame S'). The two firings happened at the same point in the train. They are colocal as observed by an observer on the train. But they occurred a distance $d = v\Delta t$ apart for an observer on earth. In Galilean relativity $\Delta t' = \Delta t$. The factor $\Gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is the relativistic correction to the classical equation. Can you tell the significance of Eqn. (2)? It is just our earlier formula for time dilation!

(C) Simultaneous Events

If the two events are simultaneous in S', then $\Delta t' = \left(t_2' - t_1'\right) = 0$ but $\Delta x' = \left(x_2' - x_1'\right) \neq 0$

$\Delta y' = 0 = \Delta z'$ say.

In frame S, the two events are described by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2). From L.T. we get

$$\Delta x = (x_2 - x_1) = \Gamma(\Delta x' + v\Delta t') = \Gamma\Delta x' \qquad \text{... (3)}$$

$$\Delta y = \Delta z = 0 \text{ but } \Delta t = \Gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = \frac{\Gamma v\Delta x'}{c^2} \neq 0 \qquad \text{... (4)}$$

Eqn. (4) shows that two events which occur at different points $\left[\left(x_1', y_1', z_1'\right) \text{ and } \left(x_2', y_2', z_2'\right)\right]$ and which are simultaneous for an observer in S', happen at two different times as observed from frame S which is in motion relative to S'. In other words, simultaneity is not an absolute property of a pair of events—it depends upon the state of motion of the observer. This may be hard to accept but it is true. In fact Eqn. (4) is the analogue of Eqn. (1). However, whereas Eqn. (1) is easy to understand, the same cannot be said of Eqn. (4). Eqns. (1) and (4) once again give a hint that space and time are more intimately related than we are accustomed to think. By the way note that Eqn. (3) is just our space contraction formula.

(D) Synchronization

Consider two inertial frames S and S' equipped with clocks located at different positions on their X-axis. Let v denote the velocity of S' relative to S. See Fig. 4.1(a). Let us see what it means to observer in S and S' when we say that clocks in S and S' are synchronized to read $t = t' = 0$ at the instant origins O and O' coincide.

Of course from the point of view of observer S , all the clocks in his frame are synchronized. However, he finds that clocks in frame S' show a time given by $t' = \Gamma\left(t - \frac{v}{c^2}x\right)$ which reduces to (when $t = 0$)

$$t' = -\Gamma \frac{v}{c^2}\ x.$$

Hence the clocks read different times depending on their location. Clocks to the left of origin (x is negative) are ahead ($t' > 0$) and those to the right are behind ($t' < 0$). Only the clock at the origin reads $t' = 0$. Hence moving clocks are not synchronized. See Fig. 4.1(b).

From the point of observer S', while clocks in his frame are synchronized at $t' = 0$, clocks in frame S, show time given by

$$t = \ \Gamma \frac{v}{c^2}\ x'.$$

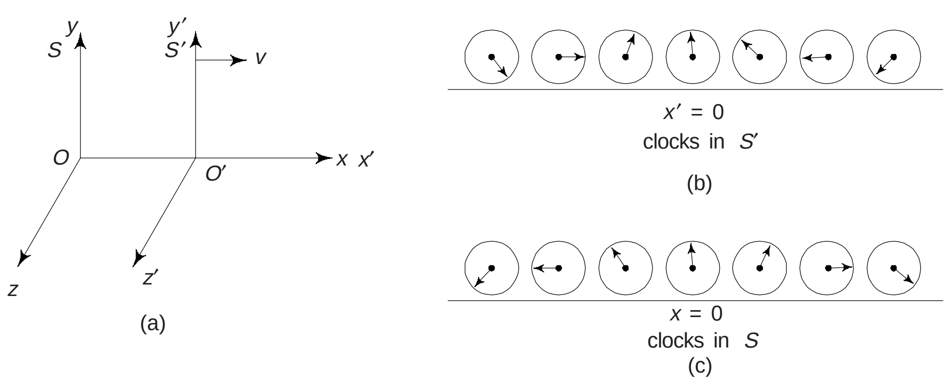

Fig. 4.1 Synchronization

Hence clocks to the left of origin (negative x) are behind, and those to the right are ahead of the central clock at the origin. See Fig. 4.1(c). Note the reciprocity in the view points of observers S and S' regarding non-synchronization of each others clocks.

4.3 PROPER TIME INTERVAL

Consider a pair of events such as the arrival of a particle in motion at two neighbouring points P and P'.

Let the space-time coordinates of the two events be (x, y, z, t) and $(x + dx, y + dy, z + dz, t + dt)$ in an inertial frame S. Let the space-time coordinates of the pair be (x', y', z', t') and $(x' + dx', y' + dy', z' + dz', t' + dt')$ in frame S'. If the frame S' (as usual in this book) is moving with a constant velocity v along the positive X-direction relative to frame S', then it follows from L. T. that

$$\left.\begin{aligned} dx' &= \Gamma(dx - vdt),\ dy' = dy,\ dz' = dz \\ dt' &= \Gamma\left(dt - \frac{vdx}{c^2}\right) \end{aligned}\right\} \qquad \text{... (1)}$$

It also follows from L.T., that

$$(dx)^2 + (dy)^2 + (dz)^2 - c^2dt^2 = (dx')^2 + (dy')^2 + (dz')^2 - c^2dt'^2 = -k^2 \text{ say.}$$

Then

$$\frac{k^2}{c^2} = \left\{dt^2 - \frac{1}{c^2}[(dx)^2 + (dy)^2 + (dz)^2]\right\}$$

$$= dt^2 \left\{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\}$$

$$= dt^2 \left(1 - \frac{u^2}{c^2}\right) \qquad \text{... (2a)}$$

where $u^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$; u being the velocity of the particle in frame S.

Similarly,

$$\frac{k^2}{c^2} = dt'^2 \left\{1 - \frac{1}{c^2}\left[\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2\right]\right\}$$

$$= dt'^2 \left(1 - \frac{u'^2}{c^2}\right) \qquad \text{... (2b)}$$

where u' is the velocity of the particle in frame S'.

From Eqns. (2a, b) we see that

$$d\tau = dt\left(1 - \frac{u^2}{c^2}\right)^{1/2} \qquad \text{... (3a)}$$

$$= dt'\left(1 - \frac{u'^2}{c^2}\right)^{1/2} \qquad \text{... (3b)}$$

From Eqns. (3a, b) we see that the quantity $d\tau$ has the same value in all inertial frames. It is therefore an invariant for all inertial observers. This invariant is called *proper time* interval.

It is seen from Eqns. (3a, b) that $d\tau = dT_0$ = time interval measured in the frame in which the particle is at rest, that is $u = 0$. Therefore proper time interval can be defined to be the time shown by a clock which is moving with the body.

Eqns. (3a, b) also shows that $dt > d\tau$; $dt' > d\tau$.

Therefore an observer with resting clocks measures a larger value for the time interval than the proper time interval—that is larger than the time interval shown by the clock moving with the body. There is nothing new here. It is just the time dilation studied earlier.

4.4 INTERVALS BETWEEN EVENTS

Consider two events described by (x, y, z, t) and $(x + dx, y + dy, z + dz, t + dt)$ in an inertial frame S. The events are separated in space and time by dx, dy, dz and dt. We know that

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2dt^2$$

is an invariant, i.e. it has the same value for all inertial observers. The (positive) square root of this quantity viz. ds is called the *interval* between the two events. Naturally ds is an invariant. In other words, a given pair of events is separated by the same interval no matter in which frame the events are observed.

Interval *ds* between two events can be imaginary, zero or real depending on $(ds)^2$ being negative, zero or positive.

(A) Timelike Interval

When $(dx)^2 + (dy)^2 + (dz)^2 < c^2dt^2$, $(ds)^2$ is negative. The interval *ds* is then imaginary. It is called *timelike interval*. In case of timelike interval, the two events are separated by such a length of time (dt), that the distance (cdt) which a ray would travel in time *dt* is greater than the space distance $[(dx)^2 + (dy)^2 + (dz)^2]^{1/2}$ between the two events. It is therefore possible to find an inertial frame S' such as a train moving with a speed $v < c$ with respect to frame *S* such that the two events appear to occur at the same place to the observer in S'. See Illustrative Example 2 of this chapter.

(B) Null Interval

When $(dx)^2 + (dy)^2 + (dz)^2 = c^2dt$, $(ds)^2$ is zero. Interval *ds* is now zero or 'null' and hence *ds* is called *null interval*.

In case of null interval, the two events are so separated in space and time that a ray of light starting from the place and time of the first event can reach the second event. Null interval is therefore also called *lightlike.*

(C) Spacelike Interval

Wen $(dx)^2 + (dy)^2 + (dz)^2 > c^2dt^2$, $(ds)^2$ is positive and *ds* is real. The interval *ds* is now called *spacelike* interval.

In case of spacelike interval, the distance in space is so large that even a ray of light cannot cover it in the available time *dt*. Only an observer or a particle or a signal travelling faster than light would have covered the distance in the available time.

Any pair of events which are simultaneous $(dt = 0)$ in an inertial frame such as *S* are separated by a spacelike interval. The interval is then spacelike for all inertial observers. Conversely if a pair of events is separated by a spacelike interval in one inertial frame, it is possible to find another inertial frame in which the events appear simultaneous.

4.5 X-t DIAGRAM, LIGHT CONE

For simplicity of understanding and visualization, let us first consider events taking place in one dimensional space, that is events on the *X*-axis. (The *y* and *z* space coordinates are just suppressed). An event specified by (x, t) is then just a point on the *X-t* diagram.

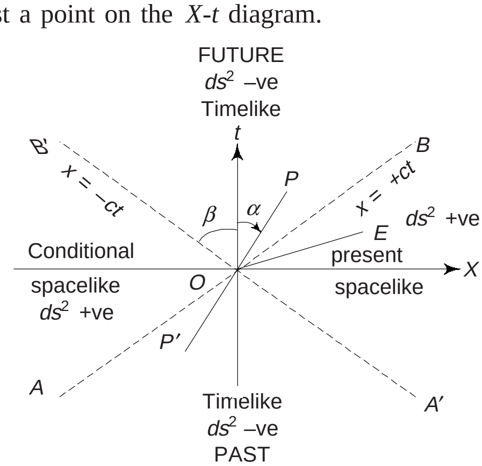

Fig. 4.2 *X-t* Diagram

Suppose a particle moves along the X-axis of a frame S with a velocity v passing through the origin O at $t = 0$. Its equation of motion is $x = vt$ and this can be graphed as a straight line $P'OP$ called as the *world line* of the particle. Every point on this graph line has a specific value of x and t—every point is thus an event. World line is just a chain of events.

For every point on the line $P'OP$, $v = x/t = \tan \alpha$. If v is negative the angle α will lie on the other side of the t axis. Since $-c < v < +c$, angle made by $P'OP$ with t axis cannot exceed β where $\tan \beta = c$. The line $P'OP$ must therefore lie between the extremes AOB and $A'OB'$. These extreme lines have equations $x = ct$ and $x = -ct$ respectively. They represent light signals travelling along the +ve and –ve X-axis respectively.

Consider the pair of events $O(0, 0)$ and $P(dx, dt)$ in the sector $B'OB$ of the diagram. The interval between these events is timelike because $(dx^2 - c^2dt^2)$ is negative. (Note that $dx = vdt$ where $v < c$). Since dt is positive, event P occurs after event O in S frame. Since the interval must be timelike in all inertial frames, there is no inertial frame in which event P occurs before event O. In other words all events represented by points in the sector $B'OB$ are absolutely in FUTURE with respect to event $O(0, 0)$.

Next, the interval between events $O(0, 0)$ and P' (say $-x, -t$) is also timelike because $(-x)^2 - c^2 (-t)^2 = (x^2 - c^2 t^2)$ is also negative. (Note that $x^2 = v^2 t^2 < c^2t^2$). Event $O(0, 0)$ occurs after event P' $(-x, -t)$ in frame S. Therefore all events represented by points in the sector AOA' are in the PAST with respect to the event O.

Consider now an event E (say X, T) in the sector BOA'. The line OE is not a possible world line for any observer (or particle) since he cannot travel faster than light. The interval between O and E is $[(x)^2 - c^2T^2]^{1/2} > 0$. The two events are separated by a spacelike interval. It is therefore possible to find a frame in which events O and E are simultaneous. (Remember the interval is positive for simultaneous events). The same is true of events in the sector AOB' . All events in these two sectors are virtually in the PRESENT relative to O. The region of these two sectors is therefore called the conditional PRESENT of O. The present is qualified as 'conditional' because it is possible to find frames in which event E occurs after event O as well as others in which event E occurs before event O. In other words the sequence (such as 'before' and 'after') of events O and E can be different in different inertial frames.

The X-t diagram can be extended by adding one more dimension viz. the Y-axis perpendicular to X-t plane. World lines of light signals leaving the origin at $t = 0$, now generate a cone whose equation is $x^2 + y^2 = c^2t^2$.

This cone divides the (x, y, t) space into past, conditional present and future regions with respect to the event $O(x = y = 0, t = 0)$. See Fig. 4.3.

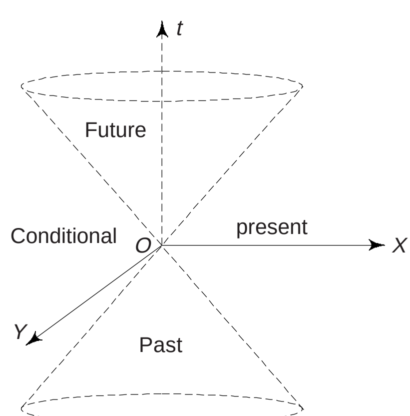

Fig. 4.3 Light Cone

Surface of the cone is made up of events which are lightlike ($ds = 0$) with respect to the event O. Hence the name *light cone.*

Interior of the light cone is made up of events which are timelike with respect to O. The set of timelike events having $t > 0$ makes up the future light cone. The set of timelike events having $t < 0$ makes up the past light cone.

In the most general case we have to imagine a four dimensional space (hyperspace) called space-time in which we have three mutually perpendicular space axis, OX, OY, OZ and a fourth time (t) axis which is perpendicular to the three space axis. In this space-time, an event is represented by giving the four coordinates (x, y, z, t).

A light hypercone with apex at the origin is now defined by the equation $x^2 + y^2 + z^2 = c^2t^2$. This divides the space-time into three regions viz. the future, the conditional present and the past in a manner similar to that in Fig. 4.3.

4.6 CAUSALITY

Two physical events are said to be causally related if one event is the effect of another preceding event which causes it. The necessary condition for two events to be causally related is that the event being caused must occur at a *later* time than the event which causes it. (This does not mean that if one event is later in time than another, it is necessarily caused by the earlier event). Now if two events are causally related in one inertial frame, then by the Principle of Relativity they must be causally related in all inertial frames. Otherwise we could distinguish between inertial frames by whether two events are causally related or not.

Suppose events E_1 (0, 0, 0, 0) and E_2 (0, 0, 0, $t > 0$) in an inertial frame are causally related. Then $ds^2 = -c^2t^2 < 0$ and the interval between then is timelike. Since ds has the same value in all inertial frames, it follows that the interval between two causally related events must be timelike in all inertial frames.

If we suppress the y and z coordinates we may refer to the X-t diagram of Fig. 4.2. We see that event O can be the cause of event P with a velocity $v < c$. Similarly P' can be the cause of event O. Note that event O can only be the cause of events which are in its future. Also O can only be the effect of events which are in its past. Thus in all inertial frames causes always precede their effects. This is the *principle of causality*.

The event E in Fig. 4.2 is spacelike with respect to event O and the two events cannot be causally related. In fact it is impossible to make a frame-independent statement such as 'earlier' or 'later' about the pair of events O and E.

If a particle or a signal could travel faster than light, then it would be possible to reach E from O. Such a particle or signal could start from O and cause the effect E (event) at a later time. This is impossible because c is the limit for signal or particle velocity. Hence the principle of causality is sometimes stated in the form: information or signal cannot travel faster than light.

SUMMARY

An event (x, y, z, t) is an occurrence which takes place at a specific place (x, y, z) at a specific time t.

Two events are coincident if they occur at the same place and at the same time. They are colocal if they occur at the same place. They are simultaneous if they occur at the same time.

If two events are coincident in one inertial frame, they are coincident in every inertial frame.

Two colocal events in S' are observed to occur a distance $\Delta x = \Gamma v \Delta \dot{t}$ apart in frame S.

Two simultaneous events in S' are separated in time by $\Delta t = \Gamma v \Delta \dot{x}/c^2$ as observed in S. Simultaneity is not an absolute property of a pair of events.

Observer in frame S finds that clocks in frame S' are not synchronized. Observer in frame S' finds that clocks in frame S are not synchronized.

Proper time interval is an invariant. It is the time shown by a clock which is moving with the body. 'Resting Clocks' show a larger value for the time interval than the time interval shown by a clock moving with the body. (Time dilation).

$ds^2 = [(dx)^2 + (dy)^2 + (dz)^2 - c^2dt^2]^{1/2}$ is called the interval between the events (x, y, z, t) and $(x + dx, y + dy, z + dz, t + dt)$. It is an invariant.

Interval between two events can be imaginary, zero or positive depending on ds^2 being negative, zero or positive. If an interval is imaginary it is called timelike, if it is zero, it is called null or lightlike and if it is positive, it is called spacelike.

If the interval between two events is timelike in a frame of reference, it is possible to find another frame in which the two events appear to occur at the same place. In case of a null interval, the two events are connected by a light signal.

If the interval between two events is spacelike, it is possible to find a reference frame in which the two events appear to be simultaneous.

An event (x, t) is specified by a point on the X-t diagram. World line is a chain of events.

World lines of all particles $(-c \le v \le c)$ lie between the light lines $x = \pm\, ct$. The region between the light lines consists of (i) events lying in the absolute future and (ii) events lying in the absolute past with respect to the event O (0, 0). The region outside is referred to as conditional present. Any event in this region cannot be specified as before or after the event $O(0, 0)$ in an absolute sense. The sequence can be different in different inertial frames.

In (X, Y, t) diagram, instead of light lines $x = \pm\, ct$, we have conical surfaces made of events which are light like with respect to the event $O(0, 0)$. In the most general case we have a hypercone which divides space-time into future, conditional present and past.

Two physical events are causally related if the earlier event causes the later event. If two events are causally related in one frame, they must be causally related in all inertial frames. Interval between two causally related events must be timelike in all inertial frames. Causes always precede their effects in every inertial frame. This is the principle of Causality.

ILLUSTRATIVE EXAMPLES

Example 1 A frame S' is moving uniformly relative to an inertial frame S along their common X-direction with a speed of $v = 0.5$ c. Identical clocks at both origins are set to zero when the origins coincide. Two simultaneous light flashes are observed in S at ($x_1 = 100$ m, $y_1 = 20$ m, $z_1 = 20$ m, $t_1 = 10^{-6}$ s) and ($x_2 = 200$ m, $y_2 = 30$ m, $z_2 = 30$ m, $t_2 = 10^{-6}$ s). At what space time coordinates are these observed in S?

Solution Since $v = 0.5$ c, $v/c = 0.5$ and $v = 0.5 \times 3 \times 10^8 = 1.5 \times 10^8$ m/s.

Also
$$\Gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{[1-(0.5)^2]^{1/2}} = \frac{1}{(0.75)^{1/2}} = 1.155$$

By L.T.,
$$x_1' = \Gamma(x_1 - vt_1) = 1.155\,[100 - (1.5 \times 10^8)10^{-6}]$$
$$= 1.155\,(-50) = -57.75 \text{ m.}$$
$$y_1' = y_1 = 20 \text{ m}; \quad z_1' = z_1 = 20 \text{ m};$$
$$t_1' = \Gamma(t_1 - vx_1/c^2) = 1.155 \left[10^{-6} - \frac{1.5\times10^8 \times 100}{9\times10^{16}}\right]$$
$$= 1.155 \left(1 - \frac{1.5}{9}\right)10^{-6} = 1.155\,(1 - 0.166)10^{-6} = 0.962 \times 10^{-6}\text{s.}$$

Similarly, $x_2' = 57.75$ m, $y_2' = 30$ m, $z_2' = 30$ m, $t_2' = 0.770 \times 10^{-6}$ s

Therefore in S' the two flashes are observed at (–57.75 m, 20 m, 20 m, 0.962×10^{-6} s) and (57.75 m, 30 m, 30 m, 0.770×10^{-6} s). The flashes are simultaneous in S but NOT simultaneous in S'.

Example 2 In an inertial frame one event occurs at $x_1 = 0$ at $t_1 = 0$ and another event occurs at $x_2 = 12$ (3×10^8) m at $t_2 = 20$ s. Show that it is possible to find an inertial frame S' in which the above two events are observed at the same place. What is the time interval between the two events in S'?

Solution In frame S, $ds^2 = (x_2 - x_1)^2 - c^2dt^2 = [(12)^2 - (20)^2]c^2 = -256c^2$. The interval between the two events is timelike. It is therefore possible to find a frame S' in which the two events occur at the same place.

Let S' move with speed v along the X-axis relative to frame S. Then

$$x_1' = \Gamma(x_1 - vt_1);\quad x_2' = \Gamma(x_2 - vt_2)$$

$$\therefore \qquad x_2' - x_1' = \Gamma[(x_2 - x_1) - v(t_2 - t_1)]$$

For events to occur at the same place in S', we must have $x_1' = x_2'$ or $(x_1' - x_2') = 0$. This will happen when

$$v = \frac{x_2 - x_1}{t_2 - t_1} = \frac{12\,c}{20} = 0.6\ c$$

Thus in frame S' moving with speed $v = 0.6$ c along the X-axis relative to frame S, the two events occur at the same place.

The factor
$$\Gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{[1 - (0.6)^2]^{1/2}} = 1.25$$

Then
$$x_1' = \Gamma(x_1 - vt_1) = 0,\quad x_2' = x_1' = 0$$

$$t_1' = \Gamma(t_1 - vx_1/c^2) = 0$$

$$t_2' = \Gamma(t_2 - vx_2/c^2) = 1.25\left(20 - \frac{0.6\,c \times 12\,c}{c^2}\right) = 1.25\ (20 - 7.2) = 16\ \text{s}$$

In S', the events are (0, 0) and (0, 16 s). In S', the time interval between the two events is 16s. This result could have been derived from time dilation or from invariance of ds^2.

Example 3 In an inertial frame S event A occurs at $x = 0$ at $t = 0$ and event B occurs at $x = 20$ c and $t = 12$ s. Find a frame S' in which the two events are simultaneous.

Solution In frame S, $ds^2 = (x_2 - x_1)^2 - c^2(t_2 - t_1)^2 = (20\ c)^2 = c^2(12)^2 = 256\ c^2$. The interval $ds = 16$ c is spacelike. Let v denote the velocity of S' relative to S. In S'

$$(t_2' - t_1') = \Gamma\lfloor(t_2 - t_1) - v(x_2 - x_1)/c^2\rfloor \text{ and the two events will be simultaneous if } (t_2' - t_1') = 0$$

That is if
$$v = \frac{c^2\ (t_2 - t_1)}{x_2 - x_1} = \frac{c^2\ (12)}{20\,c} = 0.6\ c$$

Thus in frame S' moving with velocity $v = 0.6$ c relative to frame S, the two events A and B are simultaneous.

Example 4 In an inertial frame S event A occurs at $x = 0$ at $t = 0$ and event B occurs at $x = 20$ c at $t = 12$ s. In a frame S' moving with uniform velocity $v = 4/5$ c relative to frame S, show that event B occurs earlier to event A.

Solution The factor $\Gamma = \dfrac{1}{\left(1-\dfrac{v^2}{c^2}\right)^{1/2}} = \dfrac{1}{[1-(4/5)^2]^{1/2}} = \dfrac{5}{3}$.

In S', the events A and B occur respectively at $t_1' = \Gamma\,(t_1 - vx_1/c^2) = 0$ and $t_2' = \Gamma\,(t_2 - vx_2/c^2)$

$$= \frac{5}{3}\left(12 - \frac{4c}{5}\cdot\frac{20c}{c^2}\right) = \frac{5}{3}(-4) = -\frac{20}{3}\text{ s}.$$

Thus $t_2' < t_1'$. Hence event B occurs earlier to event A in frame S'.

Example 5 An observer A is situated on the X-axis of frame S at $x = a$ and an observer B is situated on the X'-axis of S' at $x' = a$. Show that in both frames, the events (i) O passes O' and (ii) A passes B are separated by a time $\dfrac{a}{v}\left[1-\left(1-\dfrac{v^2}{c^2}\right)^{1/2}\right]$ but that the occurrence of the two events is different.

Solution Since the clocks are set (as usual in this book) so that $t = t' = 0$ when O passes O', the events are described as

(O, O) and $(x = a, t)$ in frame S

and (O, O) and $(x' = a, t')$ in frame S'.

Here t is the time noted by observer A when the observer B passes him and t' is the time noted by observer B when observer A passes him.

From L.T., $x' = a = \Gamma(x - vt) = \Gamma(a - vt)$

Or $a = \Gamma(a - vt)$ ... (1)

Similarly $t' = \left(t - vx/c^2\right)\Gamma = \Gamma\left(t - va/c^2\right)$ ... (2)

From eqn. (1), $\dfrac{a}{\Gamma} = a - vt$

$\therefore$ $t = \dfrac{a}{v}\left(1 - \dfrac{1}{\Gamma}\right) = \dfrac{a}{v}\left[1 - \left(1 - v^2/c^2\right)^{1/2}\right]$ ... (3)

Substituting this value of t in eqn. (2) and simplifying we get

$$t' = -\frac{a}{v}\left[1 - \left(1 - v^2/c^2\right)^{1/2}\right]$$

Thus though the time intervals are equal, the orders of occurrence of the two events are different. That is in frame S, the event O passes O' occurs earlier than the event A passes B because t is positive, whereas in frame S', the event A passes B occurs earlier to the event O passes O'.

Example 6 An observer notes that two events are separated in space and time by 3.6×10^8 m and 2 s. What is the proper time interval between these events?

Solution Proper time interval between two events is the time interval measured by an observer in frame say S' for whom the two events are colocal.

By L.T., $\Delta x' = \Gamma(\Delta x - v\Delta t)$ where $\Delta x = 3.6 \times 10^8$ m and $\Delta t = 2$ s

Since $\Delta x' = 0$, $v = \dfrac{\Delta x}{\Delta t} = \dfrac{3.6\times 10^8}{2} = 1.8 \times 10^8$ m/s.

$$\therefore \qquad \Gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{[1-(.6)^2]^{1/2}} = 5/4$$

Proper time $\quad \Delta t' = \Gamma\left(\Delta t - vx/c^2\right)$

$$= \frac{5}{4}\left[2 - \frac{1.8\times10^8 \times 3.6\times10^8}{9\times10^{16}}\right] = \frac{5}{4}[2 - 0.72] = 1.6 \text{ s}$$

EXERCISES

1. Explain the terms (i) event and (ii) world line.
2. What is an event? What are (i) coincident (ii) colocal and (iii) simultaneous events?
3. Discuss the relativity of (i) coincident (ii) colocal and (iii) simultaneous events.
4. Explain the concept of proper time interval? What is its relevance to dilation of time?
5. What is an interval? What are (i) Timelike (ii) Lightlike and (iii) Spacelike intervals?
6. Show that the interval between two events is invariant. (**Hint:** Use L.T.).
7. Show that if the interval between two events is timelike in one inertial frame, it is possible to find another inertial frame in which the two incidents are colocal.
8. Prove that if the interval between two events is spacelike in one inertial frame, it is possible to find another inertial frame in which the two events are simultaneous.
9. What is an X-t diagram? Show that the lines $x = \pm ct$ divide the space (x and t) into absolute future, conditional present and absolute past. What is a light cone?
10. State and explain the principle of causality.
11. Show that any two events between which a particle travels are essentially separated by a timelike interval.
12. Use the concept of interval to show that if two events are simultaneous in one inertial frame, then the spatial separation between them is less in that frame than in any other inertial frame.
13. Observer A notes that two simultaneous events occur 40 m apart. What is the time separation of this pair of events as observed by B who finds that the events occurred 50 m apart? Find his velocity relative to the observer A. (*Ans.* $\Delta t' = 10^{-7}$s, $v = 0.6$ c)
14. A particle travels at speed v for a time interval Δt as observed by an observer in the inertial frame S. Show that the proper time elapsed in the particle frame is $d\tau = \dfrac{\Delta t}{\Gamma}$. Hence find the proper time elapsed for a particle travelling at 0.99 c for a time of $\sqrt{2} \times 10^{-6}$ s. (*Ans.* 2×10^{-7} s)
15. Space-time coordinates of a pair of events in frame S are: event A (a, o, o, a/c), event B ($2a$, o, o, $a/2c$). Find the speed of frame S' in which the two events are observed to be simultaneous. When do these events occur according to the observer in S'? (*Ans.* $v = -c/2$, $t_1' = t_2' = \sqrt{3}\ a/c$)
16. Frame S' travels along the common X-X'-axis with speed $v = 0.8$ c relative to the frame S. Clocks are so set that $t = t' = 0$ when $x = x' = 0$. Two events A and B as described in frame S are: A(60 m, 10^{-7} s) and B(10 m, 2×10^{-7} s). What is the time interval between the events in frame S'? $\left(\textit{Ans.}\ \dfrac{35}{9} \times 10^{-7}\ \text{s}\right)$
17. Two events are separated in space and time by 600 m and 8×10^{-7} s as observed by an observer in frame S. Find the velocity of the frame S' relative to the frame S if the two events are simultaneous in S'. (*Ans.* 0.4 c)

18. Two colocal events in frame S are separated by a time interval of 4 seconds. What is the spatial separation between these events in frame S' in which the two events are separated by a time interval of 6 seconds? (*Ans.* 1.3416×10^9 m)
19. Two simultaneous events in frame S are separated by a distance of 1 km along the X-axis. What is the time interval between these two events as measured in frame S' in which the spatial separation between the two events is observed to be 2 km? (*Ans.* 5.77×10^{-6} s)
20. Two light bulbs in frame S situated at $x_1 = 0$ and $x_2 = 10$ km. flash simultaneously at $t = 0$. An observer in S' travelling with speed 0.6 c relative to frame S in the positive X-direction also observes the flashes. What is the time interval between the flashes according to the observer in S'? Which bulb flashes first according to him? (*Ans.* 2.5×10^{-8} s, the bulb at $x_2 = 10$ km)

Suggested Further Reading

Taylor E.F., and Wheeler J.A.: *Spacetime Physics* (Freeman, San Francisco).
Rosser W.G.C.: *An Introduction To The Theory Of Relativity* (Butterworth, London).

Chapter

5

Relativity Theory of Some Optical Effects

In this chapter we shall apply the theory of special relativity to study some simple optical phenomena.

5.1 VELOCITY OF LIGHT IN A MOVING MEDIUM (FIZEAU'S EXPERIMENTS)

In 1851 Fizeau conducted experiments to find the velocity of light in a moving medium (water). Light from a monochromatic source S is spilt into two beams by a partially silvered glass plate P. See Fig. 5.1. The transmitted beam is reflected successively by mirror M_1, M_2, M_3 and enters the telescope T after traversing the plate P. The other beam reflected at P is successively reflected by the mirrors M_3, M_2, M_1, the plate P and enters the telescope T. The two beams entering the telescope produce a steady interference pattern when the medium in the apparatus is stationary. The experiment consists of turning on the water flow and observing the fringe shift ΔN due to velocity say v of the medium.

Note that when the water flow is turned on, then the velocity of the medium (water) is along the direction of light propagation in the case of the transmitted beam (path $SPM_1M_2M_3$) and opposite to the direction of propagation in the case of beam originally reflected at plate P (path $SPM_3M_2M_1P$).

If n is the refractive index of water, the velocity of light in stationary water is c/n and hence according to Galilean theory, the expected velocity of light was $U = \left(\frac{c}{n} \pm v\right)$ depending upon whether velocity of light was in the direction of light propagation or opposite to it.

Let d denote the factor by which the velocity of the medium affects the velocity of light travelling through it. Then the velocity of light due to moving medium is $\left(\frac{c}{n} \pm vd\right)$ so that d should be equal to one if Galilean theory is correct. For certain historical reason, the factor d is called the Fresnel drag coefficient. Fizeau determined the value of d experimentally using the apparatus shown in Fig. 5.1.

Time taken by beam one (path $SPM_1M_2M_3P$) to traverse the water is $T_1 = 2l\Big/\left(\frac{c}{n} + vd\right)$ where l is the length of each water tube. The time taken by the other beam (path $SPM_3M_2M_1P$) to traverse water is $T_2 = 2l\Big/\left(\frac{c}{n} - vd\right)$.

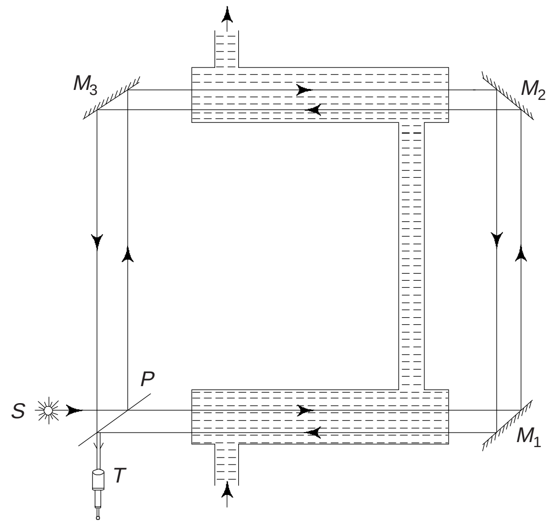

Fig. 5.1 Fizeau's Experiments

The difference between these two times is

$$(T_2 - T_1) = \Delta T = \frac{2l}{\frac{c}{n} - vd} - \frac{2l}{\frac{c}{n} + vd} = \frac{2l\left(\frac{c}{n} + vd - \frac{c}{n} + vd\right)}{(c/n)^2 - v^2 d^2}$$

$$= \frac{4lvd}{(c/n)^2 - v^2 d^2} = \frac{4lvd}{c^2/n^2} \qquad \because \quad vd \ll c/n.$$

$$\therefore \qquad \Delta T = \frac{4lvdn^2}{c^2}$$

If the time period of light used is $T = \lambda/c$, then the fringe shift ΔN when the water flow is turned on is

$$\Delta N = \frac{\Delta T}{T} = \frac{4lvdn^2}{c^2} \cdot \frac{c}{\lambda} = \frac{4lvdn^2}{c\lambda}$$

or

$$d = \left(\frac{\lambda c \Delta N}{4ln^2 v}\right)$$

Fizeau found experimentally that the drag coefficient $d = (1 - 1/n^2)$. In other words, Fizeau found that the velocity of light in moving media is not $\left(\frac{c}{n} \pm v\right)$ but $\frac{c}{n} \pm v\left(1 - \frac{1}{n^2}\right)$.

This experimental result is explained by Einstein's velocity addition formula.

In frame S' moving along with the water with velocity v, the speed of light is $U_x' = c/n$. Hence in the laboratory (frame S) the speed of light is given by

$$U_x = \frac{U_x' + v}{1 + \frac{v}{c^2} U_x'}$$

$$\therefore \qquad W = U_x = \frac{\frac{c}{n} \pm v}{1 \pm \frac{v}{c^2}\frac{c}{n}} = \frac{\frac{c}{n} \pm v}{1 \pm v/cn}$$

where the upper sign is to be taken when water flows along the direction of propagation of light and the lower sign is to be taken when water flows in the direction opposite to that of light propagation.

$$\therefore \qquad W = \frac{\frac{c}{n} \pm v}{1 \pm v/cn} = \left(\frac{c}{n} \pm v\right)\left(1 \pm \frac{v}{cn}\right)^{-1}$$

$$\approx \left(\frac{c}{n} \pm v\right)\left(1 \mp \frac{v}{cn}\right) \qquad \because \frac{v}{cn} << 1.$$

$$\approx \left(\frac{c}{n} \pm v \mp \frac{v}{n^2}\right) \text{ neglecting higher order terms in } v.$$

$$\therefore \qquad W = \frac{c}{n} \pm v\left(1 - \frac{1}{n^2}\right) \text{ in accord with the experiment.}$$

5.2 ABERRATION

Aberration is the variation in the apparent position of a heavenly body such as a star, due to the motion of the observer with the earth.

Consider a star *P* at rest in the *X*-*Y* plant of frame *S* as shown in Fig. 5.2(a). A telescope viewing the star is inclined at an angle α to the *X*-axis. Thus the position of the star is specified by angle α in frame *S* in which the star is at rest.

Let *v* denote the speed of the earth *E* along the positive *x* direction relative to frame *S*. See Fig. 5.2(b). In this frame *S*′ the telescope viewing the star is inclined at angle α_1' with the *X*′-axis.

In frame *S*, the ray *PO* has velocity components $U_x = -c \cos \alpha$, $U_y = -c \sin \alpha$ and $U_z = 0$.

According to the relativistic law of composition of velocities.

$$U_x' = \frac{U_x - v}{1 - \frac{v}{c^2} U_x} = \frac{-c \cos \alpha - v}{1 + \frac{vc \cos \alpha}{c^2}}$$

$$U_y' = \frac{U_y}{\Gamma\left(1 - \frac{v}{c^2} U_x\right)} = \frac{-c \sin \alpha \sqrt{1 - v^2/c^2}}{1 + \frac{vc \cos \alpha}{c^2}}$$

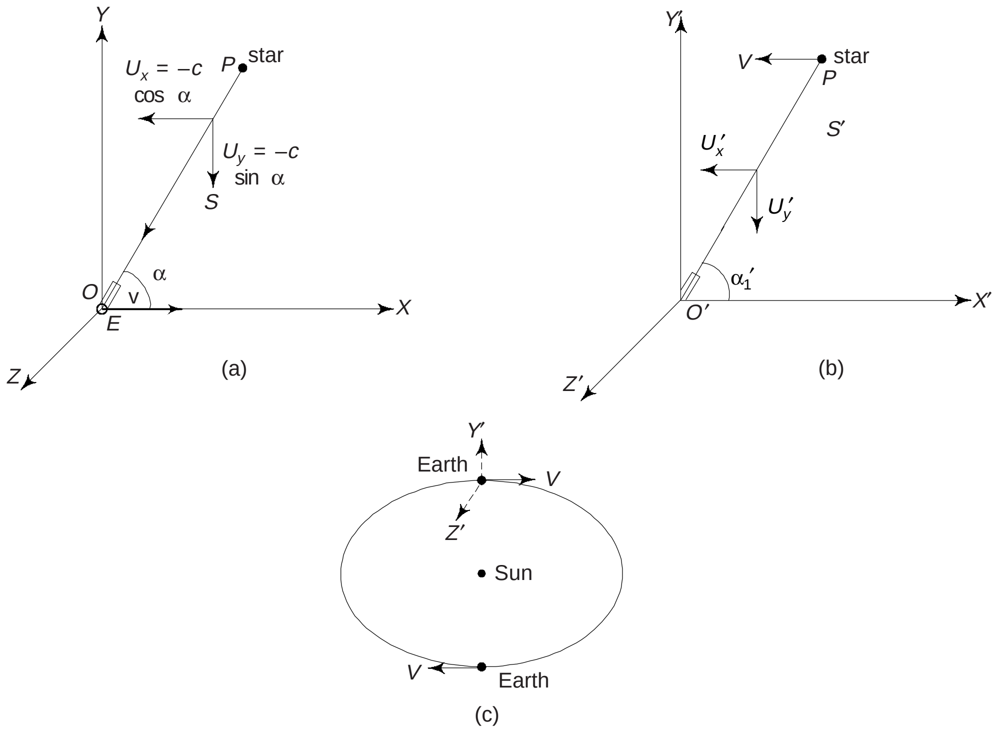

Fig. 5.2 Aberration

$$\therefore \qquad \tan\,\alpha_1' = \frac{U_y'}{U_x'} = \frac{-c\sin\alpha\sqrt{1-v^2/c^2}}{-c\cos\alpha - v} = \frac{\sin\alpha\sqrt{1-v^2/c^2}}{\cos\alpha + v/c} \qquad \text{... (1)}$$

or

$$\tan\,\alpha_1' = \frac{\tan\alpha\sqrt{1-v^2/c^2}}{1+\dfrac{v}{c}\sec\alpha} \qquad \text{... (2)}$$

Thus $\alpha_1' < \alpha$.

Six months later, velocity of earth going round the sun is oppositely directed. See Fig. 5.2(c). The star would now appear to move to the right with a velocity v. The telescope must now be inclined at angle α_2' with the X′-axis so that,

$$\tan\,\alpha_2' = \frac{\sin\alpha\sqrt{1-v^2/c^2}}{\cos\alpha - v/c} = \frac{\tan\alpha\sqrt{1-v^2/c^2}}{1-\dfrac{v}{c}\sec\alpha} \qquad \text{... (3)}$$

Eqn. (3) is obtained by replacing v by $-v$ in the previous formulae.

Note that α_2' is different from α_1'. This means that the inclination of the telescope must be changed in order to keep the star in the field of view.

Since $$v/c \approx \frac{3\times10^4}{3\times10^8} \ll 1,$$

$$\tan\ \alpha_1' = \frac{\sin\alpha}{\cos\alpha+\frac{v}{c}} = \frac{\tan\alpha}{1+\frac{v}{c}\sec\alpha} = \tan\alpha\left(1+\frac{v}{c}\sec\alpha\right)^{-1}$$

$$\approx \tan\ \alpha\left(1-\frac{v}{c}\sec\alpha\right) = \tan\ \alpha - \frac{v}{c}\ \tan\ \alpha\ \sec\ \alpha$$

$$= \tan\ \alpha - \frac{v}{c}\ \sin\ \alpha\ \sec^2\ \alpha \qquad \text{... (4)}$$

Let $\alpha_1' = \alpha + \Delta\alpha$, then

$$\tan\ \alpha_1' = \tan\ (\alpha + \Delta\alpha) \qquad \Delta\alpha \ll 1$$

$$\approx \tan\ \alpha + \Delta\alpha\ \frac{d}{d\alpha}\ (\tan\ \alpha) \qquad \text{(Taylor series)}$$

$$= \tan\ \alpha + \Delta\alpha\ \sec^2\ \alpha$$

Comparing this with Eqn. (4) we see that

$$\Delta\alpha = -\frac{v}{c}\ \sin\ \alpha \qquad \text{... (5a)}$$

$$\therefore \qquad \alpha_1' = \alpha - \frac{v}{c}\ \sin\ \alpha \qquad \text{... (6a)}$$

Thus α_1' is less than α.

Similarly it can be shown that six months later

$$\Delta\alpha = +\ \frac{v}{c}\ \sin\ \alpha \qquad \text{... (5b)}$$

or $$\alpha_2' = \alpha + \frac{v}{c}\ \sin\ \alpha \qquad \text{... (6b)}$$

Thus because of earth's motion around the sun, the inclination of the telescope has to be changed between the limits given by α_1' and α_2' so that the star remains in the field of view throughout the year.

In the case of an overhead star, the star appears to move in a circular path for which $\Delta\alpha \approx \frac{v}{c} = 10^{-4}$ rad. (See Exercise 11 of Chapter 2)

5.3 HEADLIGHT EFFECT

A moving source of radiation radiating uniformly in all directions in its rest frame, appears to radiate predominantly along its direction of motion. This observed bunching of radiation along the forward direction is due to aberration and is called as *headlight effect.*

Consider a source in frame S' radiating uniformly in all directions. Frame S' is moving along positive X-X'axis with velocity v. See Fig. 5.3(a), In frame S' the ray $O'\ P$ (in plane $X'Y'$) has velocity components $U_x' = c\cos\ \alpha'$ and $U_y' = c\sin\ \alpha'$.

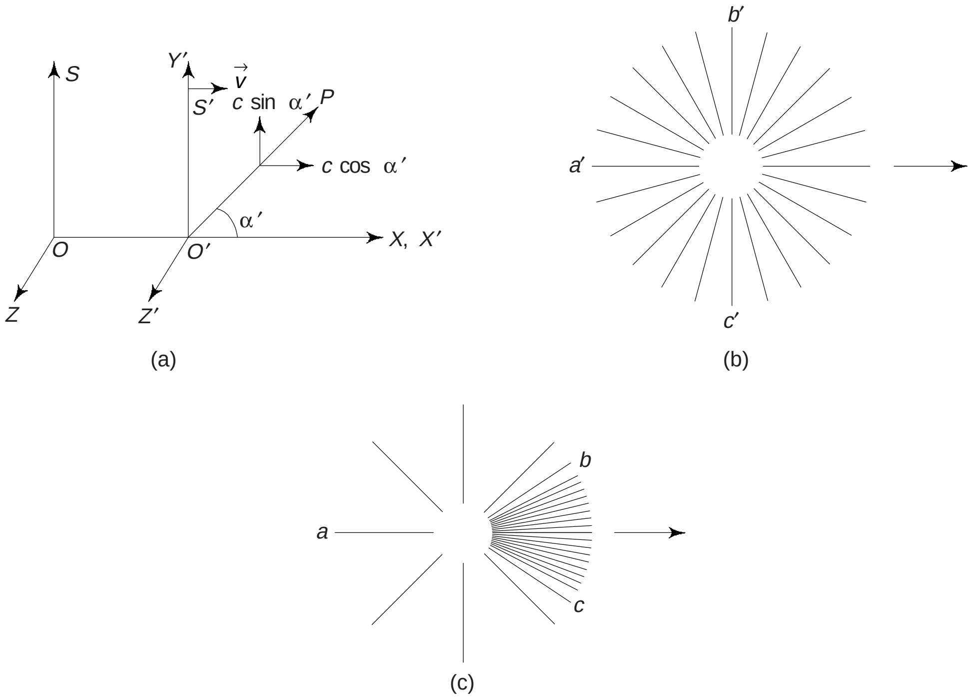

Fig. 5.3 Headlight Effect

According to the relativistic law of composition of velocities, in *S* frame the *x* component of the velocity is

$$U_x = \frac{U_x' + v}{1 + U_x' v/c^2} = \frac{c \cos \alpha' + v}{1 + c \cos \alpha' \left(\dfrac{v}{c^2}\right)} = \frac{c \cos \alpha' + v}{1 + \dfrac{v}{c} \cos \alpha'} = \frac{c(\cos \alpha' + \beta)}{1 + \beta \cos \alpha'}$$

Since $U_x = c \cos \alpha$, the angle α made by the ray with the *X*-axis as observed in frame *S* is given by

$$\cos \alpha = \frac{U_x}{c} \quad \text{or} \cos \alpha = \frac{\cos \alpha' + \beta}{1 + \beta \cos \alpha'}, \quad \text{where } \beta = \frac{v}{c}.$$

From this equation we see that $\alpha = \alpha'$ when $\alpha' = 0$ or π but $\cos \alpha = \beta$ when $\cos \alpha' = \pm \dfrac{\pi}{2}$. Then as β approaches 1, the angle α approaches zero. Thus most of the radiation appears to be strongly concentrated in the forward direction with very little radiation coming off in the backward direction. This is shown in Fig. 5.3(c) qualitatively for a source radiating uniformly in its rest frame as in Fig. 5.3(b).

The headlight effect can be observed as visible light in case of radiation emitted (synchrotron radiation) by circulating charged particles accelerated to extremely high energies in modern accelerators. A similar phenomenon is observed in nature when high energy cosmic ray protons decelerate on entering the earth's atmosphere.

5.4 DOPPLER EFFECT

Figure 5.4 shows a frame S' moving uniformly with velocity v along X-axis with respect to the frame S. The clocks in both the frames are set to zero at the instant the origins O and O' coincide.

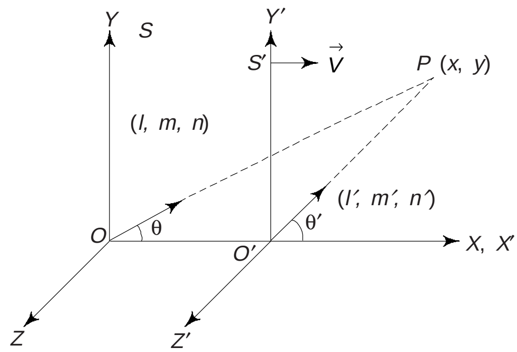

Fig. 5.4 Doppler Effect

Suppose a light source L at O' is at rest in frame S'. Assume that L emits a plane light wave of a single frequency f' as measured in frame S' when

$$x' = x = 0,\ y' = y = 0,\ z' = z = 0,\ t' = t = 0 \qquad \text{... (1)}$$

The propagation of light may be described by observers in the frames S' and S respectively by

$$\Psi' = Ae^{2\pi i\left(\frac{l'x' + m'y' + n'z'}{\lambda'} - f't'\right)} \qquad \text{... (2)}$$

$$\Psi = Ae^{2\pi i\left(\frac{lx + my + nz}{\lambda} - ft\right)} \qquad \text{... (3)}$$

Since a crest (or trough) is observed as a crest (or trough) by both observers, the phases of the two waves must be the same—they cannot differ by a multiple of 2π because then Eqn. (1) cannot be satisfied.

$$\therefore \qquad \frac{l'x' + m'y' + n'z'}{\lambda'} - f't' = \frac{lx + my + nz}{\lambda} - ft$$

Using L.T., we get

$$\frac{l'\Gamma(x - vt) + m'y' + n'z'}{\lambda'} - f'\Gamma(t - vx/c^2) = \frac{lx + my + nz}{\lambda} - ft.$$

This equality must hold for all x, y, z and t. This is possible only if the coefficients of these quantities are separately equal. Equating coefficients of x, y, z and t, we get

$$\frac{l}{\lambda} = \Gamma\left(\frac{l'}{\lambda'} + \frac{f'v}{c^2}\right);\ \frac{m}{\lambda} = \frac{m'}{\lambda'};\ \frac{n}{\lambda} = \frac{n'}{\lambda'} \qquad \text{... (4)}$$

and $$f = \Gamma\left(f' + \frac{vl'}{\lambda'}\right)$$

$\therefore$ $$f = \Gamma\left(f' + \frac{vl'f'}{c}\right)$$

or $$f = \Gamma f'\left(1 + \frac{vl'}{c}\right) \qquad \text{... (5)}$$

since $\lambda f = \lambda' f' = c$ = speed of light ... (6)

If the direction of propagation makes angles θ and θ' with the X and X' axis as observed by S and S' respectively, then $l = \cos\,\theta$ and $l' = \cos\,\theta'$.

Hence Eqn. (5) may be written as

$$f = \Gamma f'\left(1 + \frac{v}{c}\cos\theta'\right) \qquad \text{or}$$

$$f = \Gamma f'(1 + \beta\cos\theta') \qquad \text{... (7)}$$

where $\beta = v/c$.

From Eqn. (4) we see that

$$\frac{l}{\lambda} = \Gamma\left(\frac{l'}{\lambda'} + \frac{vc}{\lambda' c^2}\right) \qquad \because f' = c/\lambda'$$

$$\frac{l}{\lambda} = \frac{\Gamma l'}{\lambda'}\left(1 + \frac{v}{cl'}\right) = \frac{\Gamma l'}{\lambda'}\left(1 + \frac{\beta}{l'}\right)$$

Since $l = \cos\,\theta$ and $l' = \cos\,\theta'$, the above equation becomes

$$\cos\,\theta = \frac{\Gamma\lambda}{\lambda'}\cos\theta'\left(1 + \frac{\beta}{cos\,\theta'}\right) = \Gamma\frac{f'}{f}\cos\theta'\left(1 + \frac{\beta}{\cos\theta'}\right)$$

$$= \Gamma\frac{f'}{f}\,(\beta + \cos\,\theta')$$

$\therefore$ $$\cos\,\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \qquad \text{... (8)}$$

because $$f'/f = \frac{1}{\Gamma(1 + \beta\cos\theta')}$$ from Eqn. (7).

From Eqn. (8) we get $\cos\,\theta + \beta\cos\,\theta\cos\,\theta' = \beta + \cos\,\theta'$

$\therefore \cos\,\theta'\,(1 - \beta\cos\,\theta) = \cos\,\theta - \beta$

$\therefore$ $$\cos\,\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \qquad \text{... (9)}$$

Eqn. (7) may be now written as

$$\frac{f}{f'} = \Gamma\,(1 + \beta\cos\theta') = \Gamma\left(1 + \frac{\beta\cos\theta - \beta^2}{1 - \beta\cos\theta}\right)$$

$$= \Gamma\left(\frac{1 - \beta\cos\theta + \beta\cos\theta - \beta^2}{1 - \beta\cos\theta}\right) = \frac{\Gamma(1-\beta^2)}{1-\beta\cos\theta} = \frac{\Gamma}{\Gamma^2(1-\beta\cos\theta)}$$

$\therefore$

$$f = \frac{f'}{\Gamma(1-\beta\cos\theta)} \qquad \text{... (10)}$$

Combining Eqns. (7) and (10) we may write

$$f = \Gamma f'(1 + \beta\cos\theta') = \frac{f'}{\Gamma(1-\beta\cos\theta)} \qquad \text{... (11)}$$

which is relativistic Doppler effect formula in a very useful form.

Following points about Doppler effect are noteworthy.

(i) Consider an observer in frame S at P. When $0 \le \theta \le \pi/2$, the source of light is approaching the observer. Then $f > f'$, that is the frequency measured in frame S is higher than the proper frequency f'. This observed increase is maximum when $\theta = 0$, that is when the source is moving directly towards the observer. Then $\cos\theta = 1$ and we get

$$f = \frac{f'}{\Gamma(1-\beta)} = f'\sqrt{\frac{1+\beta}{1-\beta}} \quad \left(\because \Gamma = \frac{1}{\sqrt{1-\beta^2}}\right) \qquad \text{... (12)}$$

(ii) When $\frac{\pi}{2} \le \theta \le \pi$, we find that $f < f'$. The frequency measured in frame S is now lower than the proper frequency f'. The observed decrease is maximum when $\theta = \pi$, that is when the source is receding directly away from the observer. Then $\cos\theta = -1$ and we get

$$f = \frac{f'}{\Gamma(1+\beta)} = f'\sqrt{\frac{1-\beta}{1+\beta}} \qquad \text{... (13)}$$

Equations (12) and (13) describe the longitudinal Doppler effect which should be familiar to the student from the Doppler effect in sound. Note however, that Doppler effect in sound is not symmetrical in the motion of the source and observer. In case of light there is no distinction between source and observer moving. In other words in case of light Doppler effect is independent of source or observer motion.

(iii) There is an additional transverse Doppler effect which is not possible according to classical theory. This transverse Doppler effect occurs when $\theta = \pi/2$. Then $\cos\theta = 0$ and $f = \frac{f'}{\Gamma'}$.

Frequency measured by the observer S is less than the proper frequency f'. This is essentially due to dilation of time. See Illustrative Example 4 of Chapter 3.

SUMMARY

According to Galilean Theory the velocity of light in moving water is expected to be $\frac{c}{n} \pm v$. However, Fizeau found experimentally that the velocity of light in water is $W = c/n \pm v(1 - 1/n^2)$ where v is the velocity of water and n its refractive index. The positive sign is to be taken when light and water travel in the same direction and the negative sign is to be taken when they travel in opposite directions.

The experimentally observed velocity of light in moving water can be explained on the basis of the relativistic law of composition of velocities.

The relativistic law of composition of velocities accounts for aberration that is the variation in the apparent position of heavenly bodies due to the motion of the observer with the earth.

A moving source of radiation radiating uniformly in all directions in its rest frame appears to radiate predominantly along its direction of motion. This bunching of radiation in the forward direction is called headlight effect. It is essentially due to aberration and can be explained on the basis of law of composition of velocities.

When a source of light is approaching an observer, the observed frequency f is higher than the frequency f' as measured in a frame in which the source is at rest. On the other hand, the observed frequency f is lower when the source of light recedes away from the observer. In particular

$$f = f' \sqrt{\frac{1+\beta}{1-\beta}}$$ for source approaching directly towards observer and

$$f = f' \sqrt{\frac{1-\beta}{1+\beta}}$$ for source receding directly away from the observer.

A novel feature of relativistic Doppler effect is the transverse Doppler effect which is observed when the source is moving perpendicular to the line connecting the observer and the source: $f = f' \sqrt{1 - v^2/c^2}$. The observed frequency is less than the proper frequency f'. This is essentially on account of dilation of time.

ILLUSTRATIVE EXAMPLES

Example 1 The speed of light in still water is c/n where n is the refractive index of water. Experiments show that the speed of light in running water can be expressed as $v = \frac{c}{n} + kv$ where $k \approx 0.4$ is called the drag coefficient and v is the velocity of water. Determine the value of k using the law of addition of velocities. $n = 4/3$.

Solution

According to the law of addition of velocities,

$U_x = \dfrac{U_x' + v}{1 + \dfrac{vU_x'}{c^2}}$ where $U_x' = \dfrac{c}{n}$ is the speed of light in a frame in which the water is at rest and v is the velocity of water in a frame S in which the velocity of light is U_x.

$$U_x = \frac{\frac{c}{n}+v}{1+\frac{v}{c^2}\frac{c}{n}} = \left(\frac{c}{n}+v\right)\left(1+\frac{v}{cn}\right)^{-1}$$

$$= \left(\frac{c}{n}+v\right)\left(1-\frac{v}{cn}\right) \qquad \because \quad v/cn \ll 1$$

$$= \frac{c}{n} - \frac{cv}{cn^2} + v - \frac{v^2}{cn} = \frac{c}{n} + v\left(1-\frac{1}{n^2}\right) \text{ neglecting } v^2 \text{ terms.}$$

$$\therefore \qquad k = \left(1-\frac{1}{n^2}\right) = \left(1-\frac{9}{16}\right) = \frac{7}{16} = 0.438.$$

Example 2 How fast must you be driving your car to see a red light signal as green? Take the wavelengths of red and green lights as 6300 Å and 5400 Å respectively.

Solution In relativity only the relative motion between the source and the observer is of importance—there is no distinction between source approaching the observer and the observer approaching the source. For an observer in a car travelling directly towards a source of light with a velocity v, the source of light appears to approach directly towards him with velocity v. The observer in the car would measure a higher frequency or a lower wavelength.

Let f' be the frequency of light emitted (red signal) and f the frequency of light measured by the observer in the car. Then

$$\frac{f}{f'} = \sqrt{\frac{1+\beta}{1-\beta}} \text{ or } \frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \qquad \because \quad f'\lambda' = f\lambda = c$$

$$\therefore \qquad \frac{6300}{5400} = \sqrt{\frac{1+\beta}{1-\beta}} = \frac{7}{6}$$

$$\frac{49}{36} = \frac{1+\beta}{1-\beta}$$

$$\therefore \qquad 49 - 49\beta = 36 + 36\beta \text{ or } 85\beta = 13$$

$$\therefore \qquad \beta = \frac{13}{85} = 0.153$$

$$\therefore \qquad \frac{v}{c} = 0.153 \text{ or } v = 0.153\text{ c} = 0.153 \times 3 \times 10^8 \text{ m/s}$$

$$\therefore \qquad v = 4.59 \times 10^7 \text{ m/s.}$$

An observer in a car approaching a red signal with above velocity would see the red signal as green!

Example 3 Three identical radio transmitters A, B and C each transmitting at the frequency f_0 in its own rest frame are in motion as shown (a) What is the frequency of B's signal as received by C? (b) What is the frequency of A's signal as received by C?

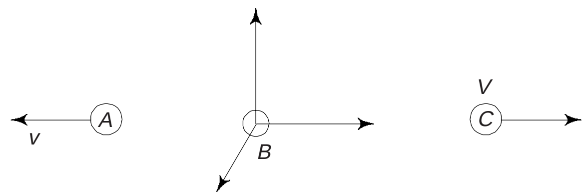

Fig. 5.5 For Illustrative Example 3

Solution

(a) For an observer on C, the source B appears to recede directly away with a velocity v. Hence the frequency observed by C is

$$f = f_0 \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \quad \text{where } \beta = v/c.$$

(b) For an observer on C, the source A appears to recede directly away with a velocity

$$v' = \frac{v+v}{1+\frac{v}{c^2}\cdot v} = \frac{2v}{1+v^2/c^2}$$

Hence the frequency of transmitter A as observed by C is given by

$$f = f_0 \left(\frac{1-v'/c}{1+v'/c}\right)^{1/2} = f_0 \left(\frac{1-\dfrac{2v/c}{1+v^2/c^2}}{1+\dfrac{2v/c}{1+v^2/c^2}}\right)^{1/2}$$

$$= f_0 \left(\frac{1+v^2/c^2-2v/c}{1+v^2/c^2+2v/c}\right)^{1/2} = f_0 \left[\frac{(1-v/c)^2}{(1+v/c)^2}\right]^{1/2} = f_0 \left(\frac{1-\beta}{1+\beta}\right)$$

Example 4 What is the Doppler shift in the wavelength of $H\alpha$ (6561 Å) line from a star which is moving away from the earth with a velocity of 300 Km/s.

Solution Observed frequency

$$f = f' \sqrt{\frac{1-\beta}{1+\beta}} \quad \text{or} \quad \frac{f}{f'} = \sqrt{\frac{1-\beta}{1+\beta}}$$

$$\therefore \qquad \frac{\lambda'}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}} \qquad (\because\ f\lambda = f'\lambda' = c)$$

$$\text{or} \qquad \lambda = \lambda' \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{where} \quad \beta = v/c = \frac{3\times 10^5}{3\times 10^8} = 10^{-3} \ll 1$$

$$\therefore \quad \sqrt{\frac{1+\beta}{1-\beta}} = (1+\beta)^{1/2}(1-\beta)^{-\frac{1}{2}} \approx (1+\beta/2)(1+\beta/2)$$

$$\approx 1 + \beta/2 + \beta/2 = 1 + \beta \text{ neglecting } \beta^2.$$

$$\therefore \quad \lambda = \lambda'(1+\beta)$$

$$\therefore \text{ Doppler shift} \quad \Delta\lambda = \lambda - \lambda' = \lambda'\beta = (6561 \text{ Å})(10^{-3})$$

$$= 6.561 \text{ Å (increase).}$$

Example 5 The sun rotates once in about 24.7 days. The radius of the sun is about 7.0×10^8 kms. Calculate the Doppler shift that we should observe for light of wavelength 6560 Å from the edge of the sun's disc near the equator. Is this shift towards the red end or the blue end of the spectrum?

Solution

The source of light at the edge *A* is instantaneously approaching the earth with a velocity $v = \omega r$ where ω is the angular velocity of the rotating sun and r its radius. The observed frequency will therefore increase, hence the observed wavelength will be less than 6560 Å. Such a decrease is called a shift towards the blue.

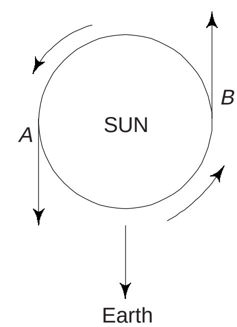

Fig. 5.6 For Illustrative Example 5

Since observed frequency $f = f'\sqrt{\frac{1+\beta}{1-\beta}}$ and $f\lambda = f\lambda' = c$,

$$\therefore \quad \frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{or} \quad \frac{\lambda}{\lambda'} = \sqrt{\frac{1-\beta}{1+\beta}} = (1-\beta)^{1/2}(1+\beta)^{-\frac{1}{2}}$$

$$\approx \left(1-\frac{\beta}{2}\right)\left(1-\frac{\beta}{2}\right) \approx 1-\beta \text{ neglecting } \beta^2 \text{ terms.}$$

$$\therefore \quad \lambda = \lambda'(1-\beta)$$

$$\therefore \quad \text{Doppler shift } \Delta\lambda = \lambda' - \lambda = \lambda'\beta$$

$$\text{Now} \quad \beta = \frac{v}{c} = \frac{\omega r}{c} = \frac{2\pi}{T}\frac{r}{c} \quad \text{where} \quad T = 24.7 \text{ days period}$$

$$\therefore \quad \Delta\lambda = \beta\lambda' = \frac{2\pi r}{Tc}\lambda' = \frac{2\pi \times 7.0 \times 10^8 \times 6560}{24.7 \times 24 \times 60 \times 60 \times 3 \times 10^8}$$

$$= 0.0455 \text{ Å}$$

$$\therefore \quad \Delta\lambda = 0.0455 \text{ Å (towards blue)}$$

The source of light at edge *B* is instantaneously receding away from the earth. The observed frequency is now reduced and hence observed wavelength will be more than 6560 Å. Such an increase in wave length is called a shift towards the red.

$$\text{In the present case} \quad \frac{\lambda}{\lambda'} = \sqrt{\frac{1+\beta}{1-\beta}} \approx (1+\beta)$$

$$\therefore \quad \lambda = \lambda'(1+\beta)$$

$$\therefore \quad \text{Increase in wavelength } \Delta\lambda = \lambda - \lambda' = \lambda'\beta$$

$$= 0.0455 \text{ Å as before (towards red)}$$

Example 6 Light of wavelength 6000 Å is incident normally on a mirror which is receding with a velocity of 3×10^4 m/s in a direction away from the incident light. Calculate the change in wavelength on reflection.

Solution

Consider an observer O moving with the mirror. To him the source of light appears to recede away directly from him. Hence the frequency of light incident on the mirror is $f = f'\sqrt{\dfrac{1-\beta}{1+\beta}}$.

This is also the frequency of light reflected from the mirror. The mirror is therefore a source of light of frequency f and this source (mirror) s receding away at 3×10^4 m/s. Hence the frequency of reflected light as observed by an observer with respect to whom the source is receding away at $v = 3 \times 10^4$ m/s is

$$f'' = f\sqrt{\frac{1-\beta}{1+\beta}} = f'\sqrt{\frac{1-\beta}{1+\beta}}\sqrt{\frac{1-\beta}{1+\beta}} = f'\left(\frac{1-\beta}{1+\beta}\right)$$

If λ'' is the observed wavelength of the reflected light then since $f''\lambda'' = c = f'\lambda'$ we find that

$$\lambda'' = \lambda'\left(\frac{1+\beta}{1-\beta}\right) = \lambda'(1+\beta)(1-\beta)^{-1} = (1+\beta)(1+\beta)$$

$$\approx \lambda'(1+2\beta)$$

$\therefore$ Change in wavelength is $\lambda'' - \lambda' = 2\beta\lambda'$

$$= 2 \times \frac{3\times 10^4}{3\times 10^8}\ 6000 = 2 \times 10^{-4} \times 6000 = 1.2\ \text{Å}.$$

EXERCISES

1. What is aberration? How is it accounted for relativistically? Discuss the aberration formula when $v \ll c$.
2. A beam of light propagates in the X–Y plane of frame S at an angle α to the X-axis. Relative to frame S', the beam is observed to make an angle α' with $O'X'$. Establish the aberration formula

   $$\cot\alpha' = \frac{\cot\alpha + \dfrac{v}{c}\operatorname{cosec}\alpha}{(1-v^2/c^2)^{1/2}}.$$
3. What is headlight effect? How is it explained by the law of addition of velocities?
4. How is the velocity of light affected by the motion of the medium in which it is travelling? Explain how the theory of relativity accounts for the experimentally observed results.
5. Describe Fizeau's experiments to determine the speed of light in a moving medium. Can the results be explained on the basis of the classical Galilean theory?
6. Give an account of the relativistic theory of Doppler effect.
7. Compare and contrast Doppler effect observed in light and sound.
8. What is Doppler effect? Show that for a source receding along the line of sight of an observer, the Doppler shift is given by $f = f_0\Gamma(1-\beta)$ where f_0 is the natural frequency and f is the observed frequency. Prove also that a source observed at right angles to the motion shows a shift of $f = f_0/\Gamma \approx f_0(1-\beta^2/2)$.
9. The phase of a light wave must be invariant under L.T. Use this fact to derive an expression for the Doppler shift in frequency of radiation emitted from a source moving with respect to an observer.

10. Isolated atoms emit light which is nearly monochromate. But light from a hot gas of atoms is found to be more diffuse in frequency when studied with the help of a spectrometer. Account for this broadening of spectral lines.
11. A galaxy has been observed to be receding away from us at a speed $v = c/2$. show that the observed wavelength is increased by a factor of $\sqrt{3}$.
12. An observer *A* sees a source of light and another observer *B* to be moving in exactly opposite directions with equal speeds of 0.6 c. Observer *A* measures the wavelength of light as 4000 Å. Calculate the natural frequency of the source and the wavelength of the source as measured by observer *B*. (*Ans.* 1.5×10^{15} Hz.; 8000 Å).
13. A beam of light of wavelength 5400 Å sent from the earth is reflected back by a rocket moving with a uniform velocity directly away from the earth. The reflected light is shifted by 1200 Å. Is this shift an increase or a decrease in wavelength? What is the velocity of the rocket?

    $\left[\textbf{Hint: } \frac{1+\beta}{1-\beta} = \frac{6600}{5400}\right]$ (*Ans.* Increase; 0.1 c)
14. The wavelength of the alpha line of hydrogen is 6560 Å. What will be the measured wavelength of the $H\alpha$ line in the absorption spectrum of a receding star moving at 3×10^6 m/s away from the earth? (*Ans.* 6625.6 Å)
15. The H_α lines measured on earth from opposite ends of the sun's equator differ in wavelength by 0.091 Å. If the solar diameter is 1.4×10^9 m, estimate the period of rotation of the solar material, assuming this rotation to be the cause of the effect. Take the wavelength of the alpha line on earth to be 6560 Å. (*Ans.* 24.7 days)
16. The wavelength of H_α line of hydrogen is 6561 Å. Calculate the width of the H_α line due to Doppler broadening at 6000 K. The maximum speed with which hydrogen atoms approach or recede from the spectrometer is given by kinetic theory of gases by the equation $\frac{1}{2} mv^2 = \frac{3}{2}$ KT. Take K = 1.38 $\times 10^{-23}$ m = 1.673×10^{-27} in S.I. units. (*Ans.* $2\lambda\beta = 0.533$ Å)
17. What is the speed of an observer approaching a star directly if he observes that half the radiation emitted from the star is bunched within a cone subtending an angle of 10^{-2} radian?
    [**Hint:** $\cos\theta = 1 - \theta^2/2 = \beta$] (*Ans.* $v = 0.9995$ c)

Suggested Further Reading

V.F. Weisskopf, *'The Visual Appearance of Rapidly Moving Objects'*, Physics Today **13**, 24(1960).

G.D. Scott and M.R. Viner, *'The Geometrical Appearance of Large Objects Moving At Relativistic Speeds'*, Am. J. Phys. **33**, 534(1965).

Rosser W.G.V.: *An Introduction To The Theory of Relativity* (Butterworth London).

A.P. French: *Special Relativity* (ELBS).

Chapter 6

Mass, Momentum and Energy

INTRODUCTION

In view of relativity of space and time, it is imperative to re-examine the concepts of mass and momentum in an inertial frame. It is then seen that these concepts undergo a radical change.

In this and the next chapter we have occasions to study collisions of two particles. Such collisions are often conveniently studied in a frame of reference in which the total linear momentum of the system is zero. The frame of reference is then called the centre of momentum frame or often a bit loosely as the centre of mass frame. In Newtonian mechanics there is no distinction between them. We shall refer to it as c. of m. frame.

In all collisions, linear momentum is conserved. Elastic collisions are those in which total kinetic energy of the system is the same before and after the collision. Collisions of ideal elastic balls in which the kinetic energy of the system is conserved is a well known example of elastic collisions. Inelastic collisions result in change of internal energies of the colliding particles as a result of which kinetic energy of the system does not remain constant. Sticking together of two particles on colliding is a well known example of inelastic collisions studied in mechanics.

6.1 VARIATION OF MASS WITH VELOCITY

According to the first postulate of relativity, all physical laws are the same in all inertial frames. Let us study the law of conservation of linear momentum in two frames S and S'. Let us assume that the mass of a particle is not a constant but is a function of the particle velocity. That is, let us suppose that the mass depends on the velocity of the body.

Suppose two identical particles moving with equal and opposite speeds as observed in frame S, collide inelastically and stick together. Since the two identical particles are moving with equal speeds we denote their mass by m when the speed is u. Momentum of the system before collision is zero ($mu - mu = 0$) and therefore the composite particle formed is at rest. Its mass is denoted by M_0. See Fig. 6.1(a).

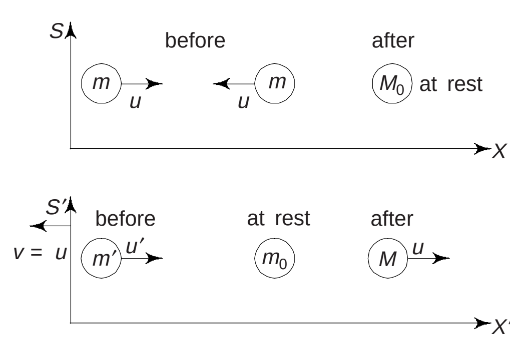

Fig. 6.1 Variation of mass with velocity

Let use now study the same collision in frame S'. In frame S' moving with velocity $v = u$ as shown in Fig. 6.1(b), one particle of mass denoted by m' moving with speed u', collides with an identical particle which is at rest (since $v = u$ as shown in the figure) and hence its mass is denoted by m_0. The composite particle which is at rest in S, is observed to move with speed u and its mass is denoted by M. If mass is independent of speed, then it would turn out that $m = m' = m_0$ and $M = M_0$.

We assume that the relativistic mass is conserved in frame S' and hence

$$M = m' + m_0 \qquad \text{... (1)}$$

By conservation of linear momentum $Mu = m'u'$ ... (2)

According to Einstein's law of addition of velocities,

$$u' = \frac{u+v}{1+uv/c^2} = \frac{2u}{1+u^2/c^2} \qquad \text{... (3)}$$

because $u = v$.

Using Eqn. (1), Eqn. (2) can be written as

$$(m' + m_0)u = m'u' \quad \text{or} \quad u = u'\left(\frac{m'}{m'+m_0}\right) \qquad \text{... (4)}$$

$$\therefore \quad u = 2u\left(\frac{m'}{1+u^2/c^2}\right)\frac{1}{m'+m_0} \quad \text{using Eqn. (3)}$$

$$\therefore \quad 2m' = (m' + m_0)(1 + u^2/c^2)$$

$$= m' + m'u^2/c^2 + m_0(1 + u^2/c^2)$$

$$\therefore \quad m'(1 - u^2/c^2) = m_0(1 + u^2/c^2)$$

$$\therefore \quad m' = \frac{m_0(1+u^2/c^2)}{(1-u^2/c^2)} = \frac{m_0}{(1-u'^2/c^2)^{1/2}} \qquad \text{... (5)}$$

because $$\left(1-\frac{u'^2}{c^2}\right) = 1 - \frac{4u^2/c^2}{(1+u^2/c^2)^2} = \frac{(1-u^2/c^2)^2}{(1+u^2/c^2)^2}$$

In Eqn. (5), m' is the mass of a particle whose speed is u'. Eqn. (5) may be written in a more convenient form as

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} \qquad \text{... (6a)}$$

for the mass of a particle travelling at speed v if its mass is m_0 when it is at rest. The particle is then said to have a *rest mass* m_0.

Particle momentum is then

$$p = mv = \frac{m_0 v}{\sqrt{1-v^2/c^2}} \qquad \text{... (7a)}$$

Note carefully that (a) c is the limiting velocity for a particle because when $v = c$; m is infinite (b) in special relativity, the momentum conserved is *not* m_0v but $p = \dfrac{m_0 v}{\sqrt{1-v^2/c^2}}$.

Eqns. (6a) and (7a) may be written respectively as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \Gamma m_0 \qquad \text{... (6b)}$$

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = \Gamma\, m_0 v = m_0\, \Gamma\, \beta c \qquad \text{... (7b)}$$

where
$$\Gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,;\ \beta = v/c \qquad \text{... (8)}$$

6.2 DYNAMICS OF A PARTICLE

(A) Concept of Force

In Newtonian mechanics, the force acting on a particle is equal to the rate of change of the particle momentum. Since mass remains constant in Newtonian mechanics, force

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\text{ (mass} \times \text{velocity)} = \text{mass } \frac{d}{dt}\text{ (velocity)}$$
$$= \text{mass} \times \text{acceleration} = m\vec{a}.$$

Hence it is also possible to define,
Force = mass × acceleration; the two definitions being equivalent.

However in relativistic mechanics,

$$\frac{d}{dt}(\vec{p}) = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

Because of the additional term $\vec{v}\,\frac{dm}{dt}$, the two definitions of force would in general be different. Experiments with high energy particles show that there is agreement between relativistic mechanics and experimental observations only if force is defined as the rate of change of momentum.

$$\vec{F} = \frac{d\vec{p}}{dt} \text{ (relativistic mechanics)} \qquad \text{... (1)}$$

(B) Concept of Work

As in Newtonian mechanics, work done dW by a force is the scalar product of the force $(\vec{F})$ acting on the particle and its displacement $d\vec{s}$. Thus

$$dW = \vec{F} \cdot d\vec{s}$$

∴ Rate of increase of (K.E.) = $\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{s}}{dt} = \vec{F} \cdot \vec{v}$ where $\vec{v}$ is the particle velocity.

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v} \qquad \text{... (2)}$$

6.3 MASS ENERGY RELATIONSHIP

Suppose a force F is applied in X-direction to a particle of rest mass m_0 initially at rest. Since the force is equal to the rate of change of momentum,

$$F = \frac{dp}{dt} = \left(\frac{dp}{dv}\right)\left(\frac{dv}{dx}\right)\left(\frac{dx}{dt}\right) = v\frac{dp}{dv}\frac{dv}{dx}$$

Work done on the particle in the course of displacement dx is

$$dW = Fdx = v\,\frac{dp}{dv}\,dv$$

Since $$p = \frac{m_0 v}{(1-v^2/c^2)^{1/2}};\ \frac{dp}{dv} = \frac{d}{dv}\left[\frac{m_0 v}{(1-v^2/c^2)^{1/2}}\right]$$

$$= \frac{m_0}{(1-v^2/c^2)^{3/2}}$$

∴ $$dW = \frac{dp}{dv}\,vdv = \frac{m_0\,vdv}{(1-v^2/c^2)^{3/2}}$$

∴ work done in accelerating the particle from rest to the final velocity v is,

$$W = \int_o^v Fdx = \int_o^v \frac{m_0\,vdv}{(1-v^2/c^2)^{3/2}} = \left[\frac{m_0\,c^2}{(1-v^2/c^2)^{1/2}}\right]_o^v$$

$$= m_0c^2\left[\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right] = c^2\left[\frac{m_0}{(1-v^2/c^2)^{1/2}} - m_0\right]$$

or $$W = c^2\,(m - m_0) = c^2\ \text{(Increase in mass)} \qquad \text{... (A)}$$

Since work done on the particle is equal to the increase in its kinetic energy, the kinetic energy of the particle is given by

$$T = c^2\,(m - m_0) = c^2\,[\Gamma m_0 - m_0]$$

∴ Kinetic Energy $$T = (\Gamma - 1)\,m_0c^2 = m_0c^2\left[\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right] \qquad \text{... (1)}$$

Since $$\Gamma = \frac{1}{(1-v^2/c^2)^{1/2}},\ (v^2/c^2) = 1 - \frac{1}{\Gamma^2} = \frac{\Gamma^2 - 1}{\Gamma^2}$$

∴ $$c^2 = \frac{\Gamma^2}{(\Gamma-1)(\Gamma+1)}\,v^2$$

∴ Substituting this in Eqn. 1, we get

Kinetic energy $$T = (\Gamma - 1)\,m_0c^2 = \frac{\Gamma^2}{\Gamma+1}\,m_0v^2 \qquad \text{... (2)}$$

In Newtonian mechanics, mass and energy are distinct physical properties. However, the relativistic expression (Eqn. A) suggests a new interpretation of mass and energy. The term (m_0c^2) has the dimensions of energy. It depends only upon the rest mass of the particle and is constant for the particle. It is independent of the velocity of the particle. In other words, *corresponding to every rest mass* m_0, *there exists energy* m_0c^2. It is called the *rest energy* or *rest mass energy* of the particle.

$$E_0 = m_0c^2 \qquad \text{... (3)}$$

Eqn. (3) implies that (just like kinetic energy) rest mass should be considered as another form of energy.

The total energy of the particle of rest mass m_0 moving with speed v relative to an inertial frame is given by

$$E = (\text{Kinetic energy } T) + (\text{rest mass energy } E_0 = m_0c^2)$$

$\therefore$ $$E = T + m_0c^2 = c^2\,(m - m_0) + m_0c^2 = mc^2 \qquad \text{... (4)}$$

or total energy $$E = (\text{Relativistic mass } m)\, c^2$$

Note also that $$E = mc^2 = \Gamma\, m_0c^2 = \Gamma E_0$$

Because rest mass can be regarded merely as another form of energy, the separate laws of mass and energy conservation of classical physics must fuse into a single conservation law in relativity, namely the *law of conservation of mass-energy*. This law holds good in any inertial system. Mass and energy are inter convertible and obviously there cannot be separate laws of conservation for mass and energy in relativistic physics.

Kinetic energy when $v \ll c$

When the particle velocity v is small compared with c ($v \ll c$), then

$$T = m_0c^2\,(\Gamma - 1) = m_0c^2\left\lfloor(1 - v^2/c^2)^{-1/2} - 1\right\rfloor$$

$$= m_0c^2\left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3v^4}{8c^4} + \ldots - 1\right] \text{ by Binomial theorem}$$

$$= \frac{1}{2}\, m_0v^2\left[1 + \frac{3}{4}\left(\frac{v}{c}\right)^2 + \ldots\right]$$

$$\approx \frac{1}{2}\, m_0v^2 \text{ when } \frac{v}{c} \ll 1.$$

Thus the relativistic expression for the kinetic energy reduces to the Newtonian result for sufficiently small velocities.

Note that as $v \to c$, the kinetic energy tends to infinity. An infinite amount of work needs to be done on a particle to accelerate it up to the speed of light c. The speed of light thus plays the role of a limiting velocity. Note also that since $T = c^2\,(m - m_0)$; change in kinetic energy of a particle is related to a change in the relativistic mass.

Some Useful Formulae

Since $$E = mc^2 \quad \text{and} \quad p = mv$$

$$E^2 - p^2c^2 = m^2c^4 - m^2v^2c^2 = m^2c^4\,(1 - v^2/c^2)$$

$$= m_0^2\,\Gamma^2c^4\,(1 - v^2/c^2) = m_0^2\,c^4$$

Thus $$E^2 = p^2c^2 + m_0^2\,c^4 \qquad \text{... (4a)}$$

or $$E^2 = (T + m_0c^2)^2 = p^2c^2 + m_0^2\,c^4$$

or $$E = (p^2c^2 + m_0^2 c^4)^{1/2} = (T + m_0c^2) \quad \text{... (4b)}$$

$\therefore$ $$T = E - m_0c^2 = [p^2c^2 + m_0^2 c^4]^{1/2} - m_0c^2 \quad \text{... (4c)}$$

From Eqn. (4b), we see that

$$E^2 = (m_0c^2 + T)^2 = p^2c^2 + m_0^2 c^4$$

$\therefore$ $$m_0^2 c^4 + T^2 + 2m_0c^2\, T = p^2c^2 + m_0^2 c^4$$

$\therefore$ $$p^2c^2 = T^2 + 2T\, m_0c^2$$

or $$p = \frac{\sqrt{T^2 + 2Tm_0 c^2}}{c} \quad \text{... (4d)}$$

Also since $$E^2 - p^2c^2 = m_0^2 c^4$$

$\therefore$ $$m_0 = \frac{\sqrt{E^2 - p^2 c^2}}{c^2} \quad \text{... (4e)}$$

If we substitute $\sin\theta = v/c$, then it is easily seen that

(a) $$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}} = m_0 \sec\theta \quad \text{... (4f)}$$

(b) Total energy $E = mc^2 = m_0c^2 \sec\theta$ ... (4g)

(c) Kinetic energy $T = (E - m_0c^2) = m_0c^2 (\sec\theta - 1)$ ... (4h)

(d) $c^2p^2 = E^2 - m_0^2 c^4 = m_0c^4 (\sec^2\theta - 1) = m_0^2 c^4 \tan^2\theta$

or $$pc = m_0c^2 \tan\theta \quad \text{... (4i)}$$

Students would find the diagram shown in Fig. 6.2 very useful.

Convenient Units

S.I. units (or for that matter, M.K.S. or C.G.S. units) are not convenient in atomic or nuclear physics where we come across particles of extremely small mass moving at extremely high velocities necessitating the use of relativistic formulae developed in this chapter. It is then convenient and more practical to work with a system of units based on the electron volt as the unit of energy.

Electron volt is the energy gained by an electron accelerated through a potential difference of 1 volt. Since the charge carried by the electron is 1.6×10^{-19} C (1.6022×10^{-19} C to be more exact) from the definition,

1 electron volt (eV) = $(1.6 \times 10^{-19}$ C) (1V)
$= 1.6 \times 10^{-19}$ J.

Other convenient units are

$$1 \text{ KeV} = 10^3 \text{ eV} = 1.6 \times 10^{-16} \text{ J};$$
$$1 \text{ MeV} = 10^6 \text{ eV} = 1.6 \times 10^{-13} \text{ J}$$

and $$1 \text{ BeV (or GeV)} = 10^9 \text{ eV} = 1.6 \times 10^{-10} \text{ J}.$$

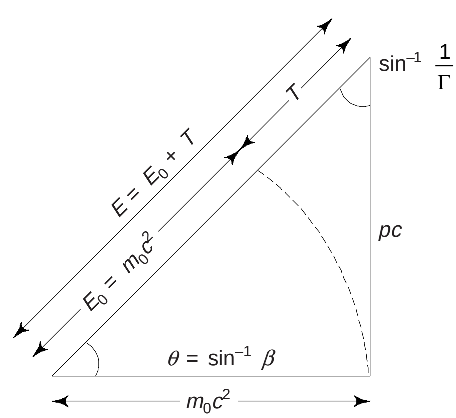

Fig. 6.2 Relations among E, T, p and m_0

For an electron, $m_0 = 9.11 \times 10^{-31}$ kg. Its rest mass energy $m_0c^2 = (9.11 \times 10^{-31})\,(3 \times 10^8)^2$ J

$$= \frac{(9.11\times10^{-31})\,(9\times10^{16})}{1.6\times10^{-13}}\ \text{MeV} = 0.511\ \text{MeV}.$$

Hence for an electron $m_0 = 0.511$ MeV/c^2

One atomic mass unit (abbreviated u) is defined as 1/12th the mass of a neutral carbon atom 6 C^{12}. Avogadro's number is $N = 6.022 \times 10^{23}$. It gives the number of atoms in 12 g of atomic carbon.

Therefore $$1\ u = \frac{1}{12}\left(\frac{12}{6.022\times10^{23}}\right)\text{g} = 1.66057 \times 10^{-27}\ \text{kg}.$$

One atomic mass unit is equivalent to energy of $(1.66 \times 10^{-27})\,(9 \times 10^{16})$ J. Since 1.60×10^{-13} J = 1 MeV, one atomic mass unit is equivalent to energy of

$$\frac{(1.66\times10^{-27})\,(9\times10^{16})}{1.60\times10^{-13}}\ \text{MeV} = 931.5\ \text{MeV}$$

$$1\ u = 931.5\ \text{MeV/C}^2$$

The student should note that often the rest mass m_0 of particles is quoted in MeV though this stands for m_0c^2 and not m_0.

Particle momentum (p) is often expressed in MeV/C because pc which represents energy is measured in MeV. Since $c = 10^8$ m/s,

$$\frac{1\ \text{MeV}}{c} = \frac{1.6\times10^{-13}\ \text{J}}{3.0\times10^{8}\ \text{m/s}} = 534 \times 10^{-22}\ \text{kg m/s}.$$

6.4 CONSERVATION OF MASS ENERGY (MASS ENERGY EQUIVALENCE)

Let us rewrite the equation,

Work done W = (Increase in mass $m - m_0$) $\times c^2$ in the form

$$(m - m_0) = \Delta m = \frac{W}{c^2}$$

This form of equation suggests that an amount of work W done on a particle results in an increase in its mass by $\Delta m = W/c^2$. There is absolutely no restriction on the source of energy added (or work done) to a system; that is inertial mass is attributed to all forms of energy—radiant energy, potential energy, thermal energy and so on. For example, mass of a body increase when it is raised to a certain height or when heat is supplied to it; mass of a spring increases when it is compressed or extended.

The idea of interconversion of mass and energy is one of the most important features of relativity. An immediate consequence of this idea is that the two laws of conservation of mass and conservation of energy get blended into one law—one may call it, the law of conservation of mass-energy. According to this law the quantity which is always conserved in any process is the total energy $E = mc^2$ which includes the rest mass energy m_0c^2. This means: any change in the internal energy (such as potential or thermal) of an isolated system causes a corresponding change $\Delta m = W/c^2$ in its rest mass energy. We take up some such instances here—Some more follow in the next chapter.

Binding Energy Of A Nucleus

Mass of a nucleus is always less than the sum of the masses of its constituent nucleons i.e., protons and neutrons. Neutrons and protons within the nucleus are held together by attractive forces. Suppose W is the work done to break a nucleus into its constituent nucleons devoid of any kinetic or potential energy. Then from the conservation of mass energy; (mass of nucleus M) $c^2 + W$ = (sum of the masses of the constituent nucleons) c^2. Here W the work done or energy supplied in breaking the nucleus is expressed in Joules and the masses should be expressed in kilograms. However nuclear masses are usually expressed in atomic mass units. As an example we take a deuteron, the nucleus of deuterium ($_1H^2$) an isotope of hydrogen. Deuteron is made up of one proton and one neutron.

Mass of proton	$m_p = 1.00728$ u
Mass of neutron	$m_n = 1.00866$ u
	$m_p + m_n = 2.01594$ u
Mass of deuteron	$m_d = 2.01355$ u

Mass which has "disappeared" in the formation of deuteron is

$$\Delta m = (2.01594 \text{ u}) - (2.01355 \text{ u}) = 0.00239 \text{ u}$$

Since 1 u is equivalent to 931.5 MeV, Δm is equivalent to (0.00239) (931.5) = 2.23 MeV. Experiments show that 2.23 MeV is indeed the energy required to break the deuteron into a proton and a neutron. W = 2.23 MeV. W is the binding energy (B.E.) of deuteron. B.E. of a nucleus is the energy which must be supplied to it in order to break it into its constituent nucleons. Another way of looking at it, is to say that the nucleons in the nucleus have a potential energy (P.E.) which is negative because of mutual attraction between nucleons. This negative potential energy reduces the mass of the nucleus by W/c^2. Note that (W/c^2) will be in kg only when W is expressed in J. In the above case, PE of the nucleons is –2.23 MeV.

Fusion and Fission Reactions

A nuclear reaction in which two lighter nuclei fuse together to form a heavier nucleus is called as fusion. In such reactions, mass of the products formed is less than the sum of the masses of the two lighter nuclei. The loss of mass is accounted for by release of energy (Q) in fusion reactions. Some such reactions are

$$_1H^2 + {_1H^2} \rightarrow {_2He^3} + n + Q = 3.3 \text{ MeV}$$
$$_1H^2 + {_1H^2} \rightarrow {_2H^3} + p + Q = 4 \text{ MeV}$$
$$_1H^2 + {_1H^3} \rightarrow {_2He^4} + n + Q = 17.6 \text{ MeV}$$

Because of (thermonuclear) fusion reactions in the interior of the sun, the sun radiates energy at the rate of about 4×10^{26} J/s. This means that our sun is losing mass at the rate of

$\Delta m = \dfrac{\Delta E}{c^2} = \dfrac{4 \times 10^{26}}{9 \times 10^{16}} \approx 4.4 \times 10^9$ kg/s. This is a loss exceeding four million tons every second.

A nuclear reaction in which a heavy nucleus breaks up into two fragments of comparable masses with a release of large amount of energy is called nuclear fission. For example when a $_{92}U^{235}$ nucleus absorbs a neutron of negligible kinetic energy, the compound nucleus $_{92}U^{236}$ formed immediately undergoes fission. In such reactions, the sum of the masses of fission products is less than the sum of masses of the original uranium nucleus and the neutron. The loss of mass is again accounted for by release of energy as in fusion reaction. Thus

$_{92}U^{235} + {_0n^1}$ (neutron) $\rightarrow {_{92}U^{236}} \rightarrow {_{54}Xe^{140}} + {_{38}Sr^{94}} + 2\,({_0n^1})$ + gamma ray ($Q \approx 200$ MeV). The energy released appears as the energy of gamma ray and kinetic energy of neutrons and fission fragments.

6.5 MASSLESS PARTICLES

According to classical electromagnetism, light is a form of wave motion which carries energy and exerts a pressure when it is incident on an object. Accordingly a pulse of radiation carrying energy E also carries a momentum $p = \dfrac{E}{c}$.

It is also known from Plancks work that radiation is emitted in the form of discrete bundles or quanta of energy. According to this quantum theory, if the frequency of radiation is ν, then the energy of each quantum commonly called a photon is given by $E = h\nu$ where h is the Planck's constant. Obviously photons travel with the speed of light c. Now according to relativity

$$E^2 = p^2c^2 + m_0^2 c^4 \text{ or } m_0 = \sqrt{(E/c^2)^2 - (p/c)^2}\ .$$

If we apply this to photons we find that since $p = E/c$, $m_0 = 0$. For this reason sometimes it is said that photons have zero rest mass. However, because photon travels with the velocity of light, it is impossible to find an inertial frame in which a photon is at rest. In other words, strictly speaking the term rest mass cannot be applied to photons.

Now for a particle of rest mass m_0, its mass, momentum and energy are given by

$$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}};\ p = \frac{m_0 v}{(1 - v^2/c^2)^{1/2}} \text{ and } E = \frac{m_0 c^2}{(1 - v^2/c^2)^{1/2}}.$$

As v tends to c, the denominator in each case tends to zero. But if the rest mass m_0 also tends to zero, then each of the above quantities can remain finite. If $m_0 \to 0$ when $v \to c$ such that $\dfrac{m_0}{(1 - v^2/c^2)^{1/2}} = k$, then $m = k$; $p = kc$ and $E = kc^2$.

To find k, we use the relation $E = h\nu$. Then $kc^2 = h\nu$ or $k = h\nu/c^2$.

Hence for photons $m = \dfrac{h\nu}{c^2}$; $p = \dfrac{h\nu}{c}$ and $E = h\nu$ ... (1)

The student may note that these relations in the form $m = E/c^2$; $p = E/c$ can be used for a neutrino of energy E. In this book we make no distinction between different types of neutrinos or their antiparticles and take their rest mass as zero.

6.6 CIRCULAR MOTION OF A CHARGED PARTICLE IN A MAGNETIC FIELD

In classical electromagnetism, the force experienced by a charged particle in an electromagnetic field is given by the so called Lorentz electromagnetic force $F = q\,(\vec{E} + \vec{v} \times \vec{B})$ where q is the charge on a particle moving with velocity $\vec{v} \cdot \vec{E}$ and $\vec{B}$ represent respectively the electric and magnetic fields in which the particle moves. The first term $q\vec{E}$ is the electrostatic force. The second term is the magnetic force

$$\vec{F}_{mag} = q(\vec{v} \times \vec{B}) \qquad \text{... (1)}$$

Newtonian mechanics needs modification on account of Einstein's theory of relativity but electromagnetic theory does not need it. Hence the magnetic force given by Eqn. (1) remains valid relativistically.

Consider a particle of rest mass m_0 and charge q moving with velocity $\vec{v}$ at right angles to a uniform magnetic field $\vec{B}$. The magnetic force is $\vec{F} = q(\vec{v} \times \vec{B})$ where $(\vec{v} \times \vec{B})$ is the vector product of $\vec{v}$ and $\vec{B} \cdot \vec{F}$ is perpendicular to both $\vec{v}$ and $\vec{B}$. In particular $\vec{F} \cdot \vec{v} = 0$.

Then
$$\vec{F} \cdot \vec{v} = \frac{dT}{dt} = \frac{d}{dt}(m - m_0)c^2$$

$$= c^2 \frac{dm}{dt} = 0$$

Hence the kinetic energy and the relativistic mass of the charged particle remains constant. Therefore the speed and the magnitude of the linear momentum also remain constant. In this situation,

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

where $\vec{a}$ = acceleration.

$$\therefore \qquad \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{m_0}{(1 - v^2/c^2)^{1/2}}\frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B})$$

This equation of motion is the same as in Newtonian mechanics *except* that the particle mass should be taken as $m = \dfrac{m_0}{(1 - v^2/c^2)^{1/2}}$ appropriate to its constant speed v.

The particle moves in a circular path of some radius say R. The symbol $\vec{a}$ represents a radial acceleration of magnitude v^2/R. Magnitude of the radial force is $F = qvB$ since $\vec{v}$ is perpendicular to $\vec{B}$.

Hence $$\frac{mv^2}{R} = qvB$$

or $$R = \frac{mv}{qB} = p/qB \qquad \text{... (2)}$$

From the measurement of the radius of circular path of a charged particle moving in a direction perpendicular to a uniform magnetic field, it is possible to determine the relativistic momentum $p = qBR$. The quantity (BR) is often called the magnetic rigidity.

For a particle carrying charge $q = ne$ where $e = 1.6 \times 10^{-19}$ C, the above equation can be written as

$$p = neBR \text{ or } pc = neBRc \text{ (J)} \qquad \text{... (3)}$$

Since $1 \text{ MeV} = 1.6 \times 10^{-13}$ J

$$pc = \frac{neBRc}{1.6 \times 10^{-13}} \text{ (MeV)} = \frac{n(1.6 \times 10^{-19})\,BR \times 3 \times 10^8}{1.6 \times 10^{-13}}$$

$$\therefore \qquad pc = 300\ nBR \text{ (MeV)} \qquad \text{... (4)}$$

or $$p = 300\ nBR \text{ (MeV/C)} \qquad \text{... (5)}$$

Note that in case B is in Gauss (and not in Tesla) and R is in centimeters (and not in metres), then p = 300 nBR (eV/C) ... (6)

6.7 EXPERIMENTAL VERIFICATION OF VARIATION OF MASS WITH VELOCITY

In 1909 A.H. Bucherer verified the variation of mass with velocity using essentially the equation $R = p/qB$ for radius of curvature (R) of a charged particle (charge q) in a uniform magnetic field (B).

Electrons (beta-rays) emitted by a radium source S pass between the plates of a large capacitor c. The capacitor consists of two parallel circular metal plates of large radius. The distance between the plates is small in comparison with the radius of the plates. The whole apparatus is enclosed in vaccum. A battery across the plates produces an electric field $\vec{E}$ along the negative Y-direction as shown in Fig. 6.3(a).

Electric force acting on the electron of (negative) charge e is Ee in the positive Y direction. A magnetic field $\vec{B}$ is applied as shown. It is perpendicular to the plane of the diagram and into the plane of the diagram.

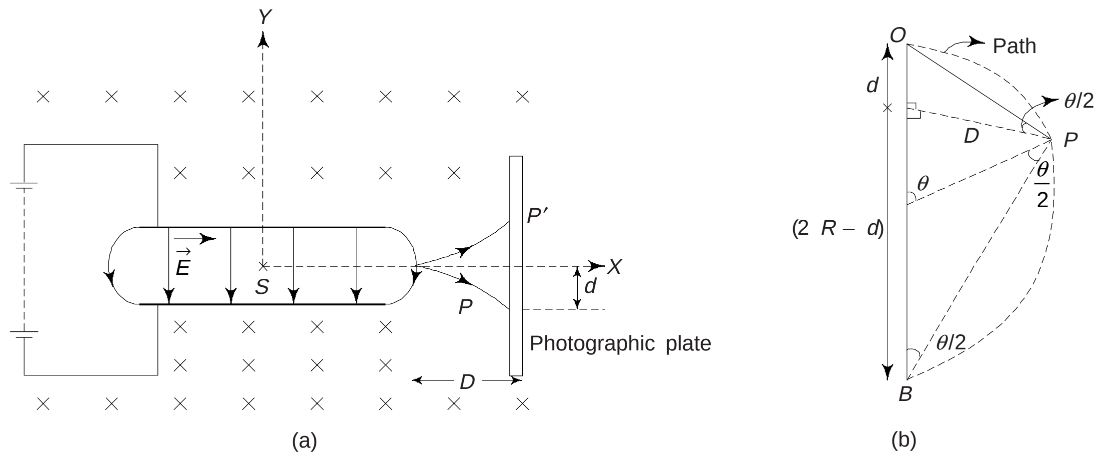

Fig. 6.3 (a) and (b) Variation of mass with velocity

The magnetic force *evB* is along the negative *Y* direction whereas the electric force is along the positive *Y* direction.

If the electric and magnetic forces are unequal, the electron would be deflected in the direction of the stronger force and fail to emerge from between the plates of the capacitor. When the two forces are exactly equal the electron will be undeflected and continue to move along *X*-axis. On coming out of the capacitor, the electron is subjected only to the magnetic force. Its path is therefore bent into a circular path of radius say *R* and it strikes the photographic plate at *P* after covering a horizontal distance *D* and a vertical distance *d* as shown. The condition for the electric and magnetic forces to be equal is

$$eE = evB \quad \text{or} \quad v = E/B \qquad \text{... (1)}$$

The semicircle on diameter *OB* in Fig. 6.3(b) is the continuation of the electron's circular path *OP*. Let θ denote the angle subtended by the chord *OP* at the centre. Then from the geometry of the figure various angles are as shown in Fig. 6.3(b). We now see that tan θ/2 = *d*/*D* and tan θ/2 is also given by *D*/(2*R* – *d*)

$$\therefore \qquad \frac{d}{D} = \frac{D}{(2R-d)}$$

$$\text{or} \qquad D^2 = d(2R - d)$$

$$\text{or} \qquad R = \frac{D^2 + d^2}{2d} \qquad \text{... (2)}$$

For motion along the circular path, momentum

$$p = mv = eBR$$

$$\therefore \qquad \frac{e}{m} = \frac{v}{BR} = \left(\frac{E}{B}\right)\frac{1}{BR} = \frac{E}{B^2}\left(\frac{2d}{D^2 + d^2}\right) \qquad \text{... (3)}$$

using Eqns. (1) and (2)

Knowing *E, B, d* and *D*; (*e*/*m*) can be calculated. Bucherer reversed electric and magnetic fields and obtained a second spot (at *P*′) on the photographic plate. He then took *d* to be equal to half the distance between the two spots at *P* and *P*′. The experiment was repeated using electrons of different velocities and (*e*/*m*) was measured for different values of *v*. The velocities were calculated using the equation *v* = *E*/*B*.

A gist of Bucherer's results is presented in the table below and Fig. 6.4

Bucherer's Results

$v = \dfrac{E}{B}$	$\dfrac{e}{m} = \dfrac{v}{RB} = \dfrac{E}{B^2}\left(\dfrac{2d}{D^2 + d^2}\right)$	$\dfrac{e}{m_0} = \dfrac{e}{m(1 - v^2/c^2)^{1/2}}$
V_1	↓Calculated values of $\dfrac{e}{m}$ decrease as v increases	Calculated values of $\dfrac{e}{m_0}$ remain practically constant
$v_2 > v_1$		
$v_3 > v_2$		
$v_4 > v_3$		
$v_5 > v_4$		
$v_6 > v_5$		

It was found that whereas the calculated values of $\dfrac{e}{m_0}$ were practically constant, $\dfrac{e}{m}$ *decreases* with increasing electron velocities. In fact the variation of mass (m) with velocity as obtained by Bucherer (X) (from second and third column of above table) fits excellently with the curve $\dfrac{m}{m_0} = \dfrac{1}{(1 - v^2/c^2)^{1/2}}$ as shown in Fig. 6.4. Solid curve is a graph of $\dfrac{1}{(1 - v^2/c^2)^{1/2}}$ versus v and crosses (x) denote $\dfrac{m}{m_0} = \dfrac{1}{(1 - v^2/c^2)^{1/2}}$ as calculated by Bucherer from his measurements.

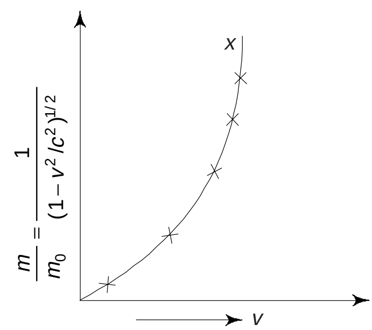

Fig. 6.4 Bucherer's results on variation of mass with velocity

Students may raise a doubt that the above experimental result can also be explained by assuming that the mass remains constant but the electron's charge varies with velocity according to $e = e_0 (1 - v^2/c^2)^{1/2}$ where e_0 is the electron's 'rest charge'. It may be said here that the relation

$m = \dfrac{m_0}{(1 - v^2/c^2)^{1/2}}$ has been verified for neutral particles such as neutrons. Electric charge is an invariant quantity (like rest mass of a particle) and this is amply proven by the success of relativistic electrodynamics based on invariance of charge.

6.8 MEASUREMENTS OF VELOCITY AND KINETIC ENERGY. BERTOZZI'S EXPERIMENTS

Bertozzi measured the speeds and kinetic energies of electrons accelerated to different energies using an ingenious combination of Van de Graaf generator and a linear accelerator. Fig. 6.5 shows a scheme of his experiments.

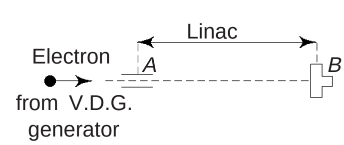

Fig. 6.5 Scheme of Bertozzi's Experiments

Electrons may be given energies upto 1.5 MeV using the Van de Graaf electrostatic generator. These electrons then enter the linear accelerator (Linac) where their energy may be further increased to as high as 15 MeV. Their speed is measured by determining the time of flight (electronically) in passing two targets separated 8.4 m apart.

The accelerated voltage is gradually varied and a graph of electron kinetic energy (T = eV) versus speed is plotted as shown in Fig. 6.6.

The relation T = eV is also checked independently. This is done by stopping the electrons in a collector at B. Their kinetic energy is converted into heat which is measured by calorimetry. Average kinetic energy per electron striking the collector is found to agree with the value eV. The experimental results are presented in Fig. 6.6 as a graph of kinetic energy versus speed.

According to Newtonian mechanics, the kinetic energy (K.E.) = $\frac{1}{2} m_0 v^2$. The experimentally observed points are seen to lie on the curve K.E. = $m_0c^2 \left[\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right] = m_0c^2\,[\Gamma - 1]$. It is also seen that the measured values of $v < c$.

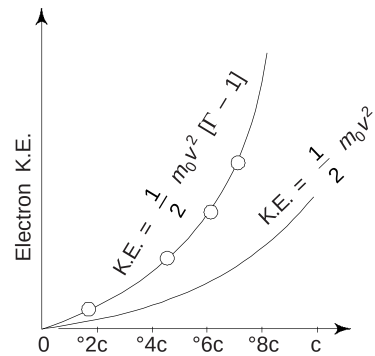

Fig. 6.6 Variation of Kinetic energy with speed

6.9 TRANSFORMATION OF MOMENTUM ENERGY AND FORCE

(A) Momentum and Energy of a Particle

Let $p_x = mu_x = \dfrac{m_0 u_x}{\sqrt{1-u^2/c^2}}$; $p_y = mu_y = \dfrac{m_0 u_y}{\sqrt{1-u^2/c^2}}$

$$p_z = mu_z = \frac{m_0 u_z}{\sqrt{1-u^2/c^2}}\ ;\ E = mc^2 = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}}$$

be respectively the x, y, z components of linear momentum and the energy of a particle as observed in a frame S.

In frame S' travelling relative to frame S at velocity v in the positive X-direction, the corresponding quantities are given by

$$p'_x = m'u'_x = \frac{m_0 u'_x}{\sqrt{1-u'^2/c^2}}\ ;\ p'_y = m'u'_y = \frac{m_0 u'_y}{\sqrt{1-u'^2/c^2}}$$

$$p'_z = m'u'_z = \frac{m_0 u'_z}{\sqrt{1-u'^2/c^2}}\ ;\ \text{and}\ \ E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}}$$

Substituting $\dfrac{1}{(1-u'^2/c^2)^{1/2}} = \dfrac{1-vu_x/c^2}{\sqrt{(1-v^2/c^2)\,(1-u^2/c^2)}}$ (See Illustrative Example 13 of Chapter 3)

and $u'_x = \dfrac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$ we get

$$p'_x = \frac{m_0\,(u_x - v)}{\sqrt{1-\dfrac{vu_x}{c^2}}} \cdot \frac{1 - vu_x/c^2}{\sqrt{(1-v^2/c^2)\,(1-u^2/c^2)}}$$

$$= \frac{m_0}{\sqrt{1-v^2/c^2}}\,\frac{(u_x - v)}{\sqrt{1-u^2/c^2}} = \Gamma\,(mu_x - mv) = \Gamma\left(p_x - \frac{Ev}{c^2}\right) \quad \text{... (1a)}$$

because $m = \dfrac{m_0}{\sqrt{1-u^2/c^2}}$, $mu_x = p_x$ and $m = E/c^2$

Similarly it can be shown that

$$p'_y = p_y \text{ and } p'_z = p_z \quad \text{... (1b)}$$

Also $E' = m'c^2 = \dfrac{m_0 c^2}{\sqrt{1-u'^2/c^2}} = \dfrac{m_0 c^2\,(1 - vu_x/c^2)}{\sqrt{(1-v^2/c^2)\,(1-u^2/c^2)}}$

$$= \Gamma mc^2\,(1 - vu_x/c^2)$$

∴ $E' = \Gamma\,(E - vp_x)$ ... (2)

Inverse relations can be immediately written down by interchanging dashed and undashed quantities and changing sign of v. They are

$$p_x = \Gamma\left(p'_x + \frac{E'v}{c^2}\right) \text{ ... (3a)} \quad p_y = p'_y \text{ ... (3b)} \quad p_z = p' \text{ ... (3c)} \quad \text{... (3abc)}$$

and $E = \Gamma\,(E' + v\,p'_x)$ ... (4)

Note that p_x, p_y, p_z and E/c transform like x, y, z and t. See L.T.

Also $p'^2 - \dfrac{E'^2}{c^2} = p'^2_x + p'^2_y - \dfrac{E'^2}{c^2} + p'^2_z$

$$= \Gamma^2\left(p_x - \frac{Ev}{c^2}\right)^2 + p_y^2 + p_z^2 - \frac{\Gamma^2}{c^2}\,(E - vp_x)^2$$

$$= p_x^2\,\Gamma^2\,(1 - v^2/c^2) + p_y^2 + p_z^2 - \frac{E^2\Gamma^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)$$

$$= p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} = p^2 - \frac{E^2}{c^2}$$

(B) Total Momentum and Energy of A Group of Particles

Consider say the ith particle from a group of particles in motion in the laboratory. Let p_{xi}, p_{yi}, p_{zi} and E_i denote its momentum components and energy respectively in the laboratory. In a frame moving along the +ve X-axis with velocity v relative to the laboratory, the above quantities are given by

$$p'_x = \Gamma\left(p_{xi} - \frac{vE_i}{c^2}\right),\ p'_{yi} = p_{yi},\ p'_{zi} = p_{zi} \text{ and}$$

$$E_i' = \Gamma(E_i - vp_{xi})$$

The total linear momentum P' and energy E' of the system of particles are obtained by summing over all the particles. Thus,

$$p'_x = \sum_i p'_{xi} = \Gamma\left(\sum_i p_{xi} - \frac{v}{c^2}\sum_i E_i\right) = \Gamma\left(P_x - \frac{vE}{c^2}\right) \quad \text{... (5a)}$$

$$P'_y = \sum_i p'_{yi} = \sum_i p_{yi} = P_y;\ P_z' = \sum_i p'_{zi} = \sum_i p_{zi} = P_z \quad \text{... (5b)}$$

and
$$E' = \sum_i E_i' = \Gamma\left(\sum E_i - v\sum_i p_{xi}\right)$$

$\therefore$
$$E' = \Gamma\,(E - vP_x) \quad \text{... (6)}$$

These equations show that *momentum and energy of a system of particles transform in the same way as in case of individual particles.*

We also note that

$$E'^2 - P'^2c^2 = E'^2 - \left(P_x'^2 + P_y'^2 + P_z'^2\right)c^2$$

$$= \Gamma^2\,(E^2 - 2EvP_x + v^2P_x^2) - c^2\left[\Gamma^2\left(P_x^2 - 2\frac{P_x vE}{c^2} + \frac{v^2E^2}{c^4}\right) + P_y^2 + P_z^2\right]$$

$$= \Gamma^2E^2\left(1 - \frac{v^2}{c^2}\right) - c^2\left[\Gamma^2P_x^2\left(1 - \frac{v^2}{c^2}\right) + P_y^2 + P_z^2\right]$$

$$= E^2 - c^2\left(P_x^2 + P_y^2 + P_z^2\right) = E^2 - P^2c^2$$

Thus
$$E'^2 - P'^2c^2 = E^2 - P^2c^2 \quad \text{... (7)}$$

(C) Transformation of Force

Let p_x, p_y, p_z, E denote respectively the x, y, z components of momentum and the energy of a particle in frame S. Let p'_x, p'_y, p'_z, E' denote the corresponding quantities in frame S'.

In frame S' force on the particle is given by $\vec{F}' = \dfrac{d\vec{p}'}{dt'}$.

$\therefore$
$$F'_x = \frac{dp'_x}{dt'} = \frac{\dfrac{dp'_x}{dt}}{\dfrac{dt'}{dt}} = \frac{\Gamma\left(\dfrac{dp_x}{dt} - \dfrac{v}{c^2}\dfrac{dE}{dt}\right)}{\Gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}$$

using L.T. and equation (1a). Thus,

$$F_x' = \frac{F_x - v/c^2 \, dE/dt}{1 - \frac{v}{c^2} u_x} \quad \text{... (8)}$$

Now since $E^2 = p^2c^2 + m_0^2 c^4 = c^2 (\vec{p} \cdot \vec{p}) + m_0^2 c^4$

$$\therefore \quad E \frac{dE}{dt} = c^2 \vec{p} \cdot \left(\frac{d\vec{p}}{dt}\right) \text{ on differentiation}$$

$$\therefore \quad mc^2 \frac{dE}{dt} = c^2 \vec{p} \cdot \left(\frac{d\vec{p}}{dt}\right)$$

$$\therefore \quad \frac{dE}{dt} = \vec{F} \cdot (\vec{p}/m) = \vec{F} \cdot \vec{u} \text{ because } \vec{F} = d\vec{p}/dt.$$

$\therefore$ Equation (8) can be written as

$$F_x' = \frac{F_x - (v/c^2)(\vec{F} \cdot \vec{u})}{1 - \frac{v}{c^2} u_x} \quad \text{... (9a)}$$

Other relations which can be similarly obtained are

$$F_y' = \frac{F_y}{\Gamma\left(1 - \frac{v}{c^2} u_x\right)} \quad \text{... (9b)}$$

and

$$F_z' = \frac{F_z}{\Gamma\left(1 - \frac{v}{c^2} u_x\right)} \quad \text{... (9c)}$$

Inverse relations obtained by changing v into $-v$ are $F_x = \dfrac{F_x' + (v/c^2)(\vec{F}' \cdot u')}{1 + v \frac{u_x'}{c^2}}$ ... (10a)

$$F'_y = \frac{F_y}{\Gamma\left(1 + \frac{v}{c^2} u_x'\right)} \quad \text{... (10b)}$$

and

$$F'_z = \frac{F_z}{\Gamma\left(1 + \frac{v}{c^2} u_x'\right)} \quad \text{... (10c)}$$

If the particle happens to be at rest in frame S, then $\vec{u} = 0$. Equations (9a), (9b) and (9c), then take the simple form

$$F_x' = F_x \quad \text{... (11a)}$$

$$F_y' = \frac{F_y}{\Gamma} \quad \text{... (11b)}$$

and

$$F_z' = \frac{F_z}{\Gamma} \quad \text{... (11c)}$$

6.10 C. OF M. FRAME

The c. of m. frame of a system of particles is defined as the inertial frame in which the total linear momentum of the system is zero. Except for a group of zero mass particles all travelling in the same direction, such a frame of reference can always be found.

Let $\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \ldots$ denote the total linear momentum of a system of particles in the laboratory frame S. We now choose the X-direction as the direction of this vector $\vec{P}$. Obviously there is now no need to use suffix x!

In frame S' moving with velocity v in the positive X-direction relative to frame S, the total momentum and energy of the system are

$$P' = \Gamma\left(P - \frac{vE}{c^2}\right) \quad \text{... (1)}$$

and

$$E' = \Gamma(E - vP) \quad \text{... (2)}$$

For the c. of m. frame, the total linear momentum P' is zero. Hence from Eqn. (1) we get

$$V = v_{\text{c.m.}} = \frac{Pc^2}{E} = \left(\frac{\text{Total momentum is S}}{\text{Total energy in S}}\right)c^2$$

Or in vector notation $\quad \vec{v}_{\text{c.m.}} = \dfrac{c^2}{E}\vec{P}$.

The analogy with Newtonian mechanics may be carried further if we imagine that there is a particle of mass M located at the c. of m., so that its momentum as observed in the laboratory frame is equal to the total linear momentum of the system of particles i.e. $\vec{P} = \Gamma M \vec{v}_{\text{c.m.}} = \dfrac{E}{c^2}\vec{v}_{\text{c.m.}}$

Or

$$M = \frac{E}{\Gamma c^2}$$

Energy of this particle in the frame S is $E = \Gamma M c^2$ = total energy of the system of particles in the frame S. Note carefully that the equivalent mass M is *not* equal to the sum of the masses of the individual particles in the system. We stress again that the equivalent particle has (i) the same momentum as the total linear momentum of the collection of particles and (ii) the same energy as the total energy of the collection of particles.

6.11 INVARIANT MASS

A quantity is said to be invariant if it is the same in different inertial frames. Rest mass of a particle is an example of an invariant quantity.

A quantity is said to be conserved if it remains unchanged in time, i.e. it remains the same after some interaction or process. For example, total linear momentum and total energy of a system are conserved quantities.

Note that an invariant quantity such as rest mass is not necessarily conserved. It may get converted into other forms of energy. Similarly a conserved quantity such as total energy or momentum of a system is not invariant since it changes from one inertial frame to another.

Let E and P denote the total energy and total linear momentum of a system of particles in one inertial frame and E' and P' the corresponding quantities in another inertial frame. Since $(E^2 - P^2c^2) = (E'^2 - P'^2c^2)$, we see that the quantity $(E^2 - P^2c^2)$ is an invariant. We can therefore write

$$(E^2 - P^2c^2) = (E'^2 - P'^2c^2) = E_0^2 - \text{Zero} = M_0^2 c^4. \qquad \text{... (1)}$$

where E_0^2 = square of the total energy E_0 of all particles in the system as measured in a frame in which the total linear momentum of the system is zero. That is, E_0 is the total energy of the system in the c. of m. frame. Eqn. (1) defines M_0 called *invariant mass of the system.* It is a quantity which is both invariant and conserved. The quantity M_0 is conserved since E and P (or E' and P') are conserved quantities. It is invariant because $(E^2 - P^2c^2) = (E'^2 - P'^2c^2)$. Note that E_0^2 is equated to $M_0^2c^4$ to make the equation dimensionally correct. The student should not construe that M_0 is the sum of the rest masses of the particles in the system. The concept of invariant mass particularly in the form $\left(E_1^2 - P_1^2 c^2\right) = \left(E_2^2 - P_2^2 c^2\right)$ can be very useful in solving problems.

SUMMARY

Linear momentum is conserved in all collisions; kinetic energy is conserved in elastic collisions but not in inelastic collisions.

The mass of a particle depends on its speed. The mass of a particle moving with speed v is given by

$m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ where m_0 (called rest mass) is its mass measured in a frame in which the particle is at rest.

Particle momentum is $p = mv = \dfrac{m_0 v}{\sqrt{1 - v^2/c^2}}$.

In Newtonian mechanics force $\vec{F} = \dfrac{d\vec{p}}{dt}$ = rate of change of momentum = $m\vec{a}$ = mass × acceleration. This is not so in relativity.

$\vec{F} = \dfrac{d\vec{p}}{dt}$ in Relativistic mechanics

As in Newtonian mechanics work done $dW = \vec{F} \cdot \overrightarrow{ds}$. If the work done is converted into kinetic energy of a particle, then $\vec{F} \cdot \vec{v} = \dfrac{dT}{dt}$ where T is the K.E. of the particle and $\vec{v}$ its velocity.

A body possesses energy by virtue of its mass. The total energy of a body is E = (Kinetic energy T) + (rest mass energy m_0c^2) = mc^2 where $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ is the relativistic mass of a body when its speed is v.

Its kinetic energy is given by

$$T = c^2\ (m - m_0) = m_0c^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right)$$

For low speeds (when $v \ll c$) $T \approx \dfrac{1}{2}\ m_0v^2$ in agreement with Newtonian mechanics.

It is convenient to measure energy in electron volts. Kinetic energy gained by an electron when accelerated through a p.d. of 1 volt is called an electron volt (eV). Masses of particles are often given in (MeV/c^2) and their momenta in (Mev/c).

The law of conservation of energy and the law of conservation of mass (matter) must be combined into a law of conservation of mass-energy. This is based on mass-energy equation. In any system, the sum of the mass and energy remains constant. This is universally applicable.

Binding energies of nuclei, energy released in fusion and fission reactions are examples of equivalence of mass and energy.

Photons of energy $h\nu$ can be treated as particles of relativistic mass $m = h\nu/c^2$ and momentum $p = h\nu/c = E/c$. Photons may be taken as having zero rest mass. This does not create any difficulty because there is no inertial frame in which a photon is at rest. Its speed is c in every inertial frame.

Charged particles (charge q) entering a uniform magnetic field ($\vec{B}$) with a velocity ($\vec{v}$) at right angles to the field are subjected to a magnetic force as a result of which the particle moves in a circular path with constant speed v. As in Newtonian mechanics, the radius of the circular path is $R = \dfrac{mv}{qB} = \dfrac{p}{qB}$. However, mass/momentum are relativistic quantities dependent on the constant speed v.

Relativistic expressions such as mass $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ and kinetic energy $T = (m - m_0)c^2 = m_0c^2$ $(\Gamma - 1)$ have been verified experimentally in case of extremely fast moving particles.

Transformation formulae for momentum components and total energy of a particle are

$$p'_x = \Gamma\left(p_x - \frac{vE}{c^2} \right);\ p'_y = p_y;\ p'_z = p_z$$

$$E' = \Gamma\ (E - vp_x)$$

For a system of particles the corresponding formulae are

$$P'_x = \Gamma\left(P_x - \frac{vE}{c^2} \right);\ P'_y = P_y;\ P'_z = P_z$$

$$E' = \Gamma(E - vP_x)$$

Also $$E'^2 - P'^2c^2 = E^2 - P^2c^2$$

C. of m. frame of a system of particles is the inertial frame S' in which the total linear momentum of the system is zero. It speed relative to frame S is

$$V_{\text{c.m.}} = \left(\frac{\text{Total momentum in } S}{\text{Total energy in } S}\right)c^2 = \frac{Pc^2}{E}$$

A quantity is called invariant if it is the same in different inertial frames. $(E^2 - P^2c^2)$ is an invariant quantity.

$(E^2 - P^2c^2) = E_0^2$ – zero = $M_0^2 c^4$ where E_0 is the total energy of the system in the c. of m. frame. The quantity M_0 defined by this equation is the invariant mass of the system.

ILLUSTRATIVE EXAMPLES

Example 1 For an extremely relativistic particle of rest energy $E_0 = m_0c^2$, show that the momentum p is given by $pc = E\left[1 - \frac{1}{2}\left(\frac{E_0}{E}\right)^2\right]$ to a good approximation.

Solution

$$E^2 = p^2c^2 + m_0^2c^4 = p^2c^2 + E_0^2$$

$\therefore$ $$p^2c^2 = E^2 - E_0^2 \text{ or } pc = (E^2 - E_0^2)^{1/2}$$

$$= E\left[1 - \left(\frac{E_0}{E}\right)^2\right]^{1/2} \approx E\left[1 - \frac{1}{2}\left(\frac{E_0}{E}\right)^2\right] \text{ when } \frac{E_0}{E} \ll 1.$$

Example 2 At what fraction of the speed of light does a particle travel if its kinetic energy is twice its rest mass energy?

Solution

$$\text{K.E.} = (m - m_0)c^2 = m_0c^2\ (\Gamma - 1) = 2\ m_0c^2$$

Or $$\Gamma = 3 \text{ or } \frac{1}{1-\beta^2} = 9$$

$\therefore$ $$1 = 9 - 9\beta^2 \text{ or } 9\beta^2 = 8 \text{ or } \beta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

$\therefore$ $$\beta = \frac{2(1.414)}{3} = \frac{2.828}{3} = 0.943.$$

Example 3 Calculate the momentum of a neutron (rest mass 940 MeV) whose kinetic energy is 200 MeV.

Solution

$$E^2 = (m_0c^2 + \text{K.E.})^2 = p^2c^2 + m_0^2c^4$$

$\therefore$ $$p^2c^2 = (m_0c^2 + \text{K.E.})^2 - m_0^2c^4$$

$$= (940 + 200)^2 - (940)^2$$

$$= (1140)^2 - (940)^2 = (2080)\ (200) = 416000$$

$\therefore \quad pc = (416000)^{1/2} = 644.9$ MeV.

$\therefore \quad p = 644.9$ MeV/C.

Example 4 What is the ratio of the relativistic mass to the rest mass for (a) an electron (b) a proton when it is accelerated from rest through a pd. of 15 megavolts. Take $m_e = 0.5$ MeV, $m_p = 1000$ MeV.

Solution

By definition of electron volt, the kinetic energy of each particle is 15 MeV.

$$\frac{m}{m_0} = \frac{mc^2}{m_0 c^2} = \frac{m_0 c^2 + \text{K.E.}}{m_0 c^2}$$

(a) $\dfrac{m}{m_0} = \dfrac{0.5 + 15}{0.5} = 31$ for electron

(b) $\dfrac{m}{m_0} = \dfrac{1000 + 15.}{1000} = 1.015$ for proton.

Example 5 Prove that the velocity of a particle can be written as $\vec{v} = \dfrac{c^2}{E}\vec{p}$ and its magnitude as $v = \dfrac{dE}{dp}$.

Solution

$$\vec{p} = m\vec{v} \quad \text{or} \quad \vec{v} = \frac{\vec{p}}{m} = \frac{\vec{p}c^2}{mc^2} = \frac{c^2}{E}\vec{p}$$

Since $\quad E^2 = p^2c^2 + m_0^2 c^4$

$$2E\frac{dE}{dp} = 2pc^2 \quad \text{or} \quad \frac{dE}{dp} = \frac{pc^2}{E} = \text{magnitude of } v.$$

Example 6 Calculate the speed of an electron (rest mass 0.5 MeV) that has been accelerated through a potential difference of 2×10^6 V (a) classically (b) relativistically. Calculate the electron mass in case (b).

(a) Kinetic energy $= \dfrac{1}{2} m_0 v^2 = \dfrac{1}{2} m_0 \beta^2 c^2 = Ve$

$$\therefore \quad \beta^2 = \frac{2Ve}{m_0 c^2} = \frac{(2)(2 \times 10^6)\ \text{eV}}{(0.5)\,10^6\ \text{eV}} = 8 \text{ or } \beta = 2\sqrt{2}$$

$\therefore \quad v = \beta c = 2\sqrt{2}\, c = 2(1.414)\ 3 \times 10^8$

$\therefore \quad v = 8.484 \times 10^8$ m/s. (This is obviously impossible).

(b) $T = m_0c^2\,(\Gamma - 1) = 2 \times 10^6$ eV $= 2$ MeV

$$\therefore \quad (\Gamma - 1) = \frac{T}{m_0 c^2} \quad \text{or} \quad \Gamma = 1 + \frac{T}{m_0 c^2}$$

$$= 1 + \frac{2}{0.5} = 5 \quad \text{or} \quad \frac{1}{(1-\beta^2)^{1/2}} = 5$$

$$\therefore \quad 1 - \beta^2 = \frac{1}{25} = 0.04$$

$$\therefore \quad \beta^2 = (1 - 0.04) \quad \text{or} \quad \beta = (1 - 0.04)^{1/2}$$

$$\therefore \quad \beta \approx 1 - \frac{1}{2}(0.04) = 0.98$$

$$\therefore \quad v = \beta c = (0.98)(3 \times 10^8) = 2.94 \times 10^8 \text{ m/s.}$$

Total electron energy $= E = mc^2 = m_0c^2 + T$

$$= (0.5 + 2) \text{ MeV} = 2.5 \text{ MeV} = 5\, m_0c^2 = mc^2$$

$$\therefore \quad m = 5\, m_0$$

Example 7 A particle has a total energy of 5 GeV and a momentum of 3 GeV/c in a certain frame of reference. (a) Find its energy in a frame in which its momentum is 4 GeV/c. (b) Calculate the rest mass of the particle.

Solution

(a) $m_0^2c^4 = E^2 - p^2c^2 = \text{constant}$

$$\therefore \quad E_2^2 - p_2^2c^2 = E_1^2 - p_1^2c^2$$

$$\therefore \quad E_2^2 = p_2^2c^2 + E_1^2 - p_1^2c^2 = (4)^2 + (5)^2 - (3)^2 = 32$$

$$\therefore \quad E_2 = \sqrt{32} \text{ GeV} = 5.656 \text{ GeV.}$$

(b) $m_0^2c^4 = E_1^2 - p_1^2c^2 = (5)^2 - (3)^2 = 16$

$$\therefore \quad m_0c^2 = 4 \text{ GeV} = 4000 \text{ MeV}$$

$$\therefore \quad m_0 = \frac{4000}{931.5} = 4.3 \text{ u.}$$

Example 8 An electron moves in a circle of 0.4 m diameter in a uniform magnetic field of 0.03 T. Obtain the speed and kinetic energy of the electron. Take $m_0c^2 = 0.511$ MeV.

Solution

$$p = \text{BeR or } pc = \text{BeRc (J)} = \text{BRc (eV)}$$

$$pc = (3 \times 10^{-2} \times 0.2 \times 3 \times 10^8) = 1.8 \times 10^6 \text{ eV} = 1.8 \text{ MeV.}$$

$$\therefore \quad E^2 = (m_0c^2 + T)^2 = p^2c^2 + m_0^2c^4$$

$$\therefore \quad (0.511 + T)^2 = (1.8)^2 + (0.511)^2$$

Solving this; kinetic energy $T = 1.360$ MeV.

Total energy $E = (1.360 + 0.511) = 1.871 \text{ MeV} = \Gamma m_0c^2$

$$\therefore \quad \Gamma = \frac{1.871}{0.511} = 3.66$$

$$\therefore \quad \Gamma^2 = \frac{1}{1-\beta^2} = (3.66)^2$$

Solving this, $\beta = v/c = 0.9618$

Or $v = 0.9618$ C.

Example 9 Calculate the radius of curvature of a proton of velocity 0.1 c in a magnetic field of 1 Wb/m^2. (For proton $m_0c^2 = 1000$ MeV).

Solution

$$R = \frac{mv}{Be} = \frac{\Gamma m_0 v}{Be} = \frac{\Gamma m_0 c^2 v}{Bec^2}$$

$$\Gamma^2 = \frac{1}{1-\beta^2} = \frac{1}{1-0.01} \quad \text{or} \quad \Gamma = 1 + \frac{1}{2}\,(0.01) = 1.005$$

$$m_0c^2 = 1000 \text{ MeV} = 1000 \times 10^6 \text{ eV} = 10^9 \text{ eV} = 10^9 \times 1.6 \times 10^{-19} \text{ J}$$

$$\therefore \quad m_0c^2 = 1.6 \times 10^{-10} \text{ J.}$$

Substituting these values

$$R = (1.005)\ \frac{1.6\times10^{-10}\times0.1\times3\times10^{8}}{(1)\,(1.6\times10^{-19})\,(9\times10^{16})} = (1.005)\ \frac{1.6\times3\times10^{-3}}{1.6\times9\times10^{-3}} = 0.335 \text{ m.}$$

Example 10 Show that the speed v of an extremely relativistic particle differs from the speed of light c by $\Delta v = c - v \approx \frac{c}{2}\left(\frac{m_0c^2}{E}\right)^2$. Find Δv for an electron of kinetic energy (a) 100 MeV, (b) 25 GeV. Take $m_0c^2 = 0.5$ MeV.

Solution

$$E^2 = m_0^2 c^4 + p^2 c^2 = m_0^2 c^4 + \frac{m_0^2 v^2 c^2}{1 - v^2/c^2} = m_0^2 c^4 + E^2\, v^2/c^2$$

$$\therefore \quad E^2\,(1 - v^2/c^2) = m_0^2 c^4$$

$$\therefore \quad E^2\,(c^2 - v^2) = m_0^2 c^6 = E^2\,(c+v)\,(c-v)$$

$$\therefore \quad \Delta v = c - v = \frac{m_0^2 c^6}{E^2\,(c+v)} = \left(\frac{m_0 c^2}{E}\right)^2 \frac{c^2}{c+v}$$

$$\approx \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2 \quad \text{when } v \approx c.$$

$$\text{(a)}\quad \Delta v = \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2 = \frac{c}{2}\left(\frac{m_0 c^2}{T + m_0 c^2}\right) = \frac{3\times10^8}{2}\left(\frac{0.5}{100.5}\right)^2$$

$$\approx \frac{3\times10^8}{2}\left(\frac{0.5}{100}\right)^2 = 1.5 \times 10^8\,(5 \times 10^{-3})^2$$

$$= 3.75 \times 10^3 \text{ m/s.}$$

(b) $\Delta v = \frac{3\times 10^8}{2}\left(\frac{0.5}{25\times 10^3 + 0.5}\right)^2 = 1.5 \times 10^8 \left(\frac{0.5}{25\times 10^3}\right)^2$

$$= 1.5 \times 10^8 (2 \times 10^{-5})^2 = 6 \times 10^{-2} = 6 \text{ cm/s.}$$

Example 11 A neutral pion moving with velocity v decays into two photons; one photon of energy E_1 travelling in the original direction of the parent pion and the other photon of energy E_2 in the exactly opposite direction. If $E_1 = 2E_2$, find v.

Solution

From conservation of energy principle,

$$\Gamma m_0 c^2 = E_1 + E_2 = 2E_2 + E_2 = 3E_2 \quad \text{... (1)}$$

From conservation of linear momentum,

$$\Gamma m_0 v = \frac{E_1}{c} - \frac{E_2}{c} = \frac{2E_2}{c} - \frac{E_2}{c} = \frac{E_2}{c} \quad \text{... (2)}$$

Dividing Eqn. (2) by Eqn. (1) we get

$$\frac{v}{c^2} = \frac{E_2/c}{3E_2} = \frac{1}{3c}$$

$\therefore$ $$v = c^2/3c = \frac{c}{3} = 10^8 \text{ m/s.}$$

Example 12 Antiprotons are produced when a beam of sufficiently fast protons strikes essentially stationary protons according to the reaction $p + p = p + p + p + \bar{P}$. Consider the collision between two protons in the c. of m. frame of reference to find the minimum energy to which the protons must be accelerated for production of antiprotons. Mass of proton/antiproton = 938 MeV.

Solution

Let E denote the total energy and p the linear momentum of the incident proton in the laboratory frame. Total energy of the system in the laboratory frame is $E_{lab} = E + m_0c^2$ where m_0c^2 = rest mass energy of the stationary proton. The total linear momentum in this frame is $p + 0 = p$.

Consider now the collision as observed in the c. of m. frame. In this frame, the total linear momentum is zero; so the two protons are observed to approach each other with equal energies say E_1 and E_2. Their momenta are equal and opposite, the total linear momentum being zero. The least energy in the c. of m. frame is that which will leave all the four product particles stationary after the reaction so that the total energy is $E' = 4\, m_0c^2 = (E_1 + E_2)$

Then $$E'^2 - (0)^2 = E_{lab}^2 - p^2c^2$$

$\therefore$ $$(4\, m_0c^2)^2 = (E + m_0c^2)^2 - p^2c^2$$

$$= E^2 - p^2c^2 + m_0^2c^4 + 2Em_0c^2$$

$$= m_0^2c^4 + m_0^2c^4 + 2Em_0c^2$$

$\therefore$ $$16\, m_0^2c^4 = 2\, m_0^2c^4 + 2Em_0c^2$$

$\therefore$ $$E = \frac{14\, m_0^2c^4}{2\, m_0c^2} = 7\, m_0c^2$$

Of this total energy of the incident proton, m_0c^2 is provided by its rest mass. Hence the minimum energy (kinetic energy) provided by the accelerator is 6 m_0c^2 = 6(938 MeV) = 5628 MeV.

Example 13 A particle of speed v is incident on another identical stationary particle. Show that the c. of m. of the system travels with a velocity given by $v_{\text{c.m.}} = \dfrac{\Gamma v}{1+\Gamma}$. Also show that $\Gamma_{\text{c.m.}} = \left(\dfrac{\Gamma+1}{2}\right)^{1/2}$.

Solution

Total linear momentum of the system is $\Gamma\, m_0 v$ and the total energy of the system is $(\Gamma\, m_0c^2 + m_0c^2) = (\Gamma + 1)m_0c^2$ where m_0 is the rest mass of each particle. The velocity of c. of m. is

$$v_{c.} = \frac{(\text{Total momentum})\,c^2}{\text{Total energy}} = \frac{\Gamma m_0\, v c^2}{(\Gamma+1)\, m_0\, c^2} = \frac{\Gamma v}{\Gamma+1}$$

$$\Gamma^2_{\text{c.m.}} = \frac{1}{1-\dfrac{v^2_{\text{c.m.}}}{c^2}} = \frac{1}{1-\dfrac{\Gamma^2 v^2}{c^2\,(\Gamma+1)^2}} = \frac{1}{1-\dfrac{\Gamma^2\beta^2}{(\Gamma+1)^2}} = \frac{(\Gamma+1)^2}{(\Gamma+1)^2-\Gamma^2\beta^2}$$

Now
$$\Gamma^2\,\beta^2 = \frac{v^2/c^2}{1-v^2/c^2} = \frac{v^2/c^2-1+1}{1-v^2/c^2}$$

$$= -1 + \frac{1}{1-v^2/c^2} = \Gamma^2 - 1$$

$$\Gamma^2_{\text{c.m.}} = \frac{(\Gamma+1)^2}{(\Gamma+1)^2-\Gamma^2\beta^2} = \frac{(\Gamma+1)^2}{(\Gamma+1)^2-(\Gamma^2-1)} = \frac{(\Gamma+1)^2}{2(\Gamma+1)} = \frac{\Gamma+1}{2}$$

∴
$$\Gamma_{\text{c.m.}} = \left(\frac{\Gamma+1}{2}\right)^{1/2}.$$

Example 14 Momentum of a particle is given to be m_0c. What is (i) its speed. (ii) its mass (iii) its kinetic energy?

Solution

(i) $m_0\Gamma v = m_0c$ or $v = \dfrac{c}{\Gamma}$ or $v^2 = \dfrac{c^2}{\Gamma^2}$

∴ $v^2 = c^2\,(1 - v^2/c^2) = c^2 - v^2$ or $2v^2 = c^2$ or $v = \dfrac{c}{\sqrt{2}}$

∴ $v = \dfrac{c\sqrt{2}}{2} = \dfrac{c\,(1.414)}{2}$ = 0.707 C.

(ii) Mass $m = m_0\Gamma = m_0\,\dfrac{c}{v} = \dfrac{m_0\, c\sqrt{2}}{c} = 1.414\; m_0$

(iii) Kinetic energy $T = (\Gamma - 1)\ m_0c^2 = (1.414 - 1)\ m_0c^2$
$= 0.414\ m_0c^2$.

Example 15 What is the radius of curvature of a 100 MeV electron in a magnetic field of 10000 Gauss? For electron $m_0c^2 = 0.51$ MeV.

Solution

$$B = 10000 \text{ G} = 1 \text{ T}.$$

$$P^2c^2 = E^2 - m_0^2 c^4 = 100^2 - 0.51^2 \approx 100^2$$

Or $$pc \approx E \approx 100 \text{ MeV}$$

$$R(m) = \frac{pc\,(\text{MeV})}{300\,\text{nB}} = \frac{100}{300\,(1)(1)} = 0.33 \text{ m}.$$

Example 16 Compute the radius of curvature of a proton of velocity 0.1 c in a magnetic field of 1 T. Take $m_0c^2 = 10^3$ MeV.

Solution

$$\Gamma = (1 - 0.01)^{-1/2} = 1 + \frac{1}{2}\,(0.01) = 1.005$$

$$R = \frac{pc}{300\,\text{nB}} = \frac{\Gamma m_0\, vc}{300\,\text{nB}} = \frac{(1.005)\, m_0\, c^2\,(0.1)}{300\,\text{nB}}$$

$$= \frac{(1.005)\,(10^3)\,(0.1)}{300\,(1)(1)} = 0.335 \text{ m}.$$

EXERCISES

1. Distinguish between elastic and inelastic collisions.
2. Derive the formula describing variation of mass with velocity.
3. Two formulae for the force are given below. Which one is suitable for use in relativistic mechanics? Why? (a) $\vec{F} = \dfrac{d\vec{p}}{dt}$ (b) $\vec{F} = m\vec{a}$.
4. Prove that the kinetic energy of a particle can be written as $T = (\Gamma - 1)\ m_0c^2$. Explain the notation.
5. Show that at low speeds the relativistic kinetic energy reduces to Newtonian expression.
6. Write down the mass energy equivalence formula and derive it.
7. Assuming that $E^2 = (T + m_0c^2)^2 = (pc)^2 + (m_0c^2)^2$
   (a) Derive the following relationships
   (i) $T = c\sqrt{m_0^2 c^2 + p^2} - m_0c^2$
   (ii) $p = \dfrac{\sqrt{T^2 + 2\,m_0\,c^2\,T}}{c}$
   (iii) $m_o = \dfrac{\sqrt{E^2 - p^2 c^2}}{c^2}$

(b) Let $\beta = \dfrac{v}{c} = \sin\theta$. Show that

(i) $E = m_0c^2 \sec\theta$

(ii) $T = m_0c^2 (\sec\theta - 1)$

(iii) $p^2c^2 = m_0^2 c^4 (\sec^2\theta - 1)$

8. Show that $T = \left(\dfrac{\Gamma^2}{1+\Gamma}\right) m_0v^2 = \dfrac{p^2}{(1+\Gamma)\, m_0}$.
9. Explain the concept of nuclear binding energy.
10. What are photons? Can they be called massless particles? Explain.
11. Derive an expression for the radius of the circular path of a fast moving charged particle in a magnetic field. In what way does it differ from the expressions for a particle moving at low speed?
12. Describe in brief Bucherer's experiments to test the relation $m = \dfrac{m_0}{(1 - v^2/c^2)^{1/2}}$.
13. Give a brief account of Bertozzi's experiments on measurements of speeds and kinetic energies of electrons? What is the conclusion drawn?
14. Write down the formulae for transformation of momenta and energy and derive them.
15. Distinguish between an invariant quantity and a conserved quantity. Show that $E^2 - p^2c^2 = E'^2 - p'^2c^2$.
16. An electron (m_0c^2 = 0.511 MeV) has a velocity of 2.4×10^8 m/s. How much energy will it lose in being slowed down to 1.8×10^8 m/s?
    [**Hint:** $m_0c^2\ (\Gamma_1 - \Gamma_2)$.] (*Ans.* 0.213 MeV)
17. Show that the rest mass of a particle of momentum p and kinetic energy T is given by $m_o = \dfrac{p^2c^2 - T^2}{2Tc^2}$. Calculate the rest mass of a particle which has a momentum of 130 MeV/C when its kinetic energy is 50 MeV. (1 MeV = 1.6×10^{-13} J).
    [**Hint:** $E^2 = (T + m_0c^2)^2 = (p^2c^2 + m_0^2c^4)$; m_0c^2 = 144 MeV.] (*Ans.* $m_0 = 25.6 \times 10^{-29}$ kg)
18. Calculate the K.E. of a neutron (rest mass 940 MeV) if its velocity is 0.8 c.
    (*Ans.* 626.6 MeV).
19. An electron (E = 0.8 MeV) moves in a uniform magnetic field in a circular path of radius 0.05 m. What is the magnetic flux density? (*Ans.* 0.041 T)
20. A charged particle $\left(\beta = \dfrac{v}{c} = \dfrac{1}{\sqrt{2}}\right)$ is observed to travel in a circular path of radius 0.46 m under the influence of a magnetic field of 1 T. The charged particle is known to carry a charge of 1.6×10^{-19} C. Find its mass in units of electron rest mass. Take m_ec^2 = 0.5 MeV.
    [**Hint:** $n\Gamma\beta\ m_ec^2$ = BRC (eV).] (*Ans.* 276 m_e)
21. Determine the radius of curvature of an electron in a magnetic field of 1 Wb/m^2 when its speed is given by (a) β = 0.3; (b) β = 0.99; (c) β = 0.999. Take m_ec^2 = 0.5 MeV.
    (*Ans.* (a) 5.225×10^{-4} m; (b) 1.17 cm; (c) 3.723 cm)
22. What is the radius of curvature of a proton (m_0c^2 = 1000 MeV) of velocity 3×10^7 m/s in a magnetic field of 1 T? (*Ans.* 0.335 m)
23. At what fraction of the speed of light does a particle travel if its kinetic energy is (a) equal to its rest mass energy (b) twice its rest mass energy? (*Ans.* (a) 0.866; (b) 0.943)

24. Calculate the velocity of a neutron (rest mass 940 MeV) of kinetic energy 200 MeV. (*Ans.* 0.566 C)
25. Calculate the radius of the circular path of a 20 MeV electron (rest mass 0.5 MeV/c^2) travelling in a uniform magnetic field of 5 T. $\left(\textit{Ans.}\ \frac{4}{3}\ \text{cm}\right)$
26. What is the velocity of a particle whose momentum is m_0c? Express its kinetic energy T and its total energy E in units of m_0c^2. $\left(\textit{Ans.}\ v = \frac{c}{\sqrt{2}},\ T = \left(\sqrt{2}-1\right)\ m_0c^2,\ E = \sqrt{2}\ m_0c^2\right)$
27. A particle has a total energy of 5 GeV and a linear momentum of 3 GeV/C in a frame S_1. What is its energy in a frame S_2 in which its momentum is 4 GeV/c? What is its rest mass in atomic mass units? What is the relative velocity of the two frames S_1 and S_2?

   $\left[\textbf{Hint: } \Gamma_1 = 1.25;\ \Gamma_2 = \sqrt{2};\ v_1 = 0.6\text{ c};\ v_2 = \frac{c}{\sqrt{2}}.\right]$ (*Ans.* $4\sqrt{2}$ GeV; 4.3 amu; 0.186 C)
28. Show that a relativistic particle of kinetic energy T and rest mass m_0 has a momentum p given by $pc = (T^2 + 2Tm_0c^2)^{1/2}$. Show that this gives $p = m_0v$ when $T \ll m_0c^2$. Determine pc in units of kinetic energy T when $\frac{T}{m_0c^2}$ = (a) 0.020 (b) 0.20 (c) 2.00 and (d) 20.

   $\left[\textbf{Hint: } pc = T\left(1+\frac{2}{T/m_0c^2}\right)^{1/2}.\right]$ (*Ans.* (a) $T(101)^{1/2}$; (b) $T(11)^{1/2}$; (c) $T\sqrt{2}$; (d) $T(1.1)^{1/2}$)
29. For a particle of rest mass m_o, kinetic energy T and momentum p, show that $\frac{p\beta c}{T} = \frac{\frac{T}{m_0c^2}+2}{\frac{T}{m_0c^2}+1}$. What is the significance of the two extreme cases when (a) T $\ll m_0c^2$ and (b) T $\gg m_0c^2$.

   $\left[\textbf{Hint: } \frac{T}{m_0c^2} = \Gamma - 1;\ \text{R.H.S.} = \frac{\Gamma+1}{\Gamma};\ \beta^2 = 1 - \frac{1}{\Gamma^2}.\right]$ $\left(\textit{Ans.}\ \text{(a)}\ T \approx \frac{m_0v^2}{2};\ \text{(b)}\ T \approx p\beta c\right)$
30. An electron (rest mass energy 0.511 MeV) has a velocity of $c/3$. Calculate the loss in its energy when decelerated to half of its velocity. (*Ans.* 0.024 MeV)
31. Find the kinetic energy of a neutron (rest mass 942 MeV) if its velocity is 0.8 C. (*Ans.* 628 MeV)
32. Calculate the momentum of a neutron (rest mass 940 MeV) if its kinetic energy is 200 MeV. (*Ans.* 645 MeV/C)
33. A proton (rest mass 940 MeV) has a momentum of 200 MeV/C. Find its kinetic energy. (*Ans.* 21.0 MeV)
34. What is the mass of a particle (rest mass 1 GeV) when its kinetic energy is 1 GeV? (*Ans.* 2 GeV/C^2)
35. Find the velocity of an electron (mass 0.5 MeV) when accelerated through a p.d. of (a) 10^4 V (b) 10^5 V and (c) 10^6 V. Calculate m/m_0 in each case.

   (*Ans.* (a) 0.2 c, 1.02; (b) 0.553 c, 1.2; (c) 0.943 c, 3)

36. Find the velocity of a neutron (rest mass energy 1000 MeV) when its kinetic energy is (a) 100 MeV; (b) 1000 MeV and (c) 10000 MeV. (*Ans.* (a) 0.42 c; (b) 0.87 c; (c) 0.999 c)
37. Calculate the amount of work in MeV required to impart an electron (rest mass 0.5 MeV) initially at rest, a velocity of (a) $\frac{c}{2}$; (b) $\frac{3c}{4}$. Find the ratio m/m_0 in each case.

    (*Ans.* (a) 0.076; 1.15 (b) 0.25; 1.51)
38. A neutral pion travelling at a velocity v decays into two photons; one of energy E_1 travelling in the original direction of the parent pion and the other of energy E_2 in the opposite direction. If $E_1 = 3E_2$ show that $v = c/2$.

Suggested Further Reading

W. Bertozzi, "*Speed And Kinetic Energy of Relativistic Electrons*" Am. J. Phys. **32**, 551(1964).
R.T. Weidner, "*On Weighing Photons*" Am. J. Phys. **35**, 443(1967).
Rosser W.G.V., *An Introduction To The Theory of Relativity*, (Butterworth London).
A.P. French, *Special Relativity* (ELBS).
Smith J.H.: *Introduction to Special Relativity* (Benjamin, New York).
Bohm D.: *The Special Theory of Relativity* (Benjamin, New York, 1962).

Chapter 7

Relativistic Mechanics

INTRODUCTION

In this chapter we study a number of topics such as decay of particles, absorption/emission of light by atoms, pair production, annihilation of matter, different types of collisions etc. Central to the study of these topics here are two important principles viz. conservation of linear momentum and conservation of total energy inclusive of mass. In other words, we study only the 'mechanics' of the topics. This study will give the students a feel of relativity at work.

7.1 PARTICLE DECAY

Because of their intrinsic instability a number of particles are known to break up or decay into two or more particles. The simplest example of such a decay is that of a particle at rest decaying into two particles.

(A) Decay of A Particle At Rest

Suppose a particle (rest mass M) at rest (momentum $p = 0$) decays into two particles of rest masses M_1 and M_2 as shown in Fig. 7.1.

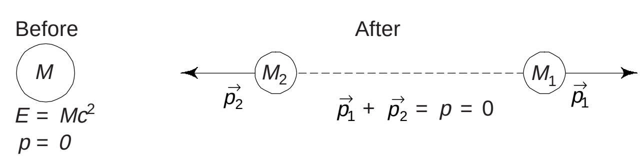

Fig. 7.1 Decay of a particle at rest

Since the initial momentum is zero, the two particles M_1 and M_2 must fly apart with equal and opposite momenta $\vec{p}_1$ and $\vec{p}_2$ respectively. Thus in order to conserve momentum, p_1 and p_2 must be equal in magnitude i.e. $p_1 = p_2$ and opposite in direction. If E_1 and E_2 represent the total energies of particles of rest masses M_1 and M_2 respectively, then we must have

$$p_1^2 c^2 = p_2^2 c^2 \quad \text{or} \quad E_1^2 - M_1^2 c^4 = E_2^2 - M_2^2 c^4$$

$$\therefore \qquad E_1^2 - E_2^2 = M_1^2 c^4 - M_2^2 c^4 \qquad \text{... (1)}$$

From conservation of total energy, we have

$$E_1 + E_2 = Mc^2 = \text{initial total energy} \qquad \text{... (2)}$$

Dividing Eqn. (1) by Eqn. (2) we get

$$E_1 - E_2 = \frac{M_1^2 c^4 - M_2^2 c^4}{Mc^2} \qquad \text{... (3)}$$

Addition and subtraction of Eqns. (2) and (3) leads respectively to

$$E_1 = \frac{\left(M^2 + M_1^2 - M_2^2\right)c^2}{2M} \qquad \text{... (4)}$$

And

$$E_2 = \frac{\left(M^2 + M_2^2 - M_1^2\right)c^2}{2M} \qquad \text{... (5)}$$

In this decay process, part of the original rest mass energy Mc^2 has been converted into sum of rest mass energies $(M_1 + M_2)c^2$ of decay products. The remaining energy $[M - (M_1 + M_2)]c^2$ appears as the kinetic energy of the decay products.

(B) Decay of A Particle In Flight

Suppose a particle of rest mass M moving with velocity $\vec{v}$ relative to frame S decays into two particles of rest mass M_1 and M_2. See Fig. 7.2(a).

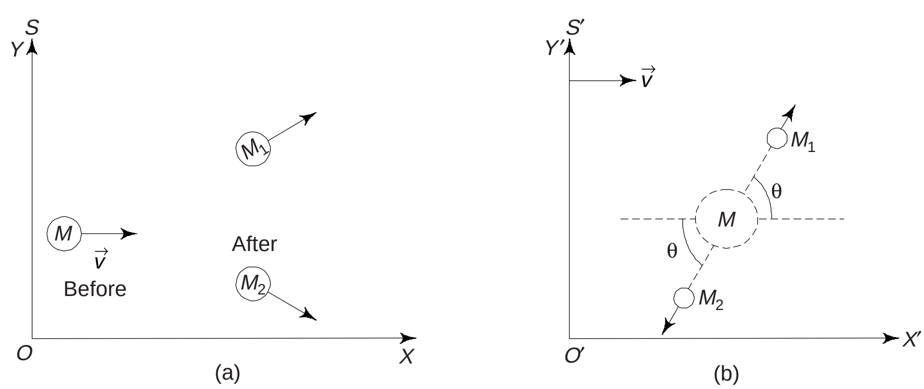

Fig. 7.2 Decay of a particle in flight

In frame S' moving to the right with velocity $\vec{v}$ as shown in Fig. 7.2(b), the particle of rest mass M is at rest. It decays into two particles of rest mass M_1 and M_2 as shown.

Using Eqns. (4) and (5) above, the total energies of particles emitted are (note that these quantities are now measured in S') given by

$$E'_1 = \frac{\left(M^2 + M_1^2 - M_2^2\right)c^2}{2M} \qquad \text{... (6)}$$

$$E'_2 = \frac{\left(M^2 + M_2^2 - M_1^2\right)c^2}{2M} \qquad \text{... (7)}$$

Their momenta are given by

$$p_1'c = p_2'c = \sqrt{E_1^2 - M_1^2c^4} = \sqrt{E_2^2 - M_2^2c^4} \quad \text{... (8)}$$

Momenta and energies of the particles in frame S are then given by equations of transformations studied in last chapter. For a simple application of the results of this article ($\theta = 0$) see Illustrative Examples.

7.2 ABSORPTION AND EMISSION OF RADIATION

(A) Absorption of Radiation

Suppose a photon of energy Q and momentum Q/c is incident on an atom (molecule, nucleus) of rest mass M_0 and is completely absorbed by it. In order to conserve momentum, after absorption of photon, the atom must recoil as shown in Fig. 7.3. Let M' be its mass and v its velocity. Then from conservation of linear momentum,

$$\frac{Q}{c} = M'v \quad \text{... (1)}$$

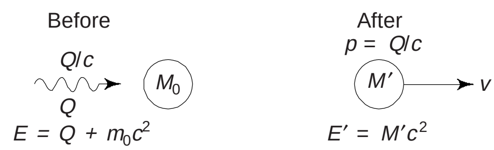

Fig. 7.3 Absorption of photon

Total energy before absorption ($Q + M_0c^2$) must be equal to the total energy after absorption $M'c^2$.

$$\therefore \quad Q + M_0c^2 = M'c^2 \quad \text{... (2)}$$

From Eqn. (2), we get $M' = M_0 + Q/c^2$ ... (3)

From Eqn. (1), we get

$$v = \frac{Q}{cM'} = \frac{Q}{c(M_0 + Q/c^2)} \quad \text{... (4a)}$$

or

$$\beta = \frac{v}{c} = \frac{Q}{Q + M_0c^2} \quad \text{... (4b)}$$

Eqns. (4a, b) give the velocity of the atom on absorbing the photon.

(B) Emission of Radiation

When an atom in an excited state emits a photon, the atom must recoil in order to conserve linear momentum. This is similar to the recoiling of a gun when a bullet is fired from it. Due to the kinetic energy of recoil, the energy of the photon emitted is less than the excitation energy possessed by the atom when it was in the excited state.

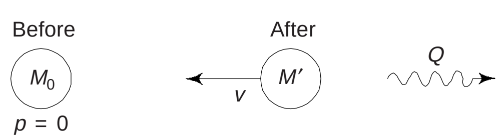

Fig. 7.4 Emission of photon

Suppose a stationary atom ($p = 0$) of rest mass M_0 emits a photon of energy Q and recoils with a velocity v. Let the recoiling atom have a mass M' and a rest mass M'_0.

For conservation of linear momentum,

$$p = 0 = M'v - \frac{Q}{c} = p' \text{ (say)} - \frac{Q}{c} \quad \text{... (5)}$$

For conservation of total energy, initial total energy (M_0c^2) = Final total energy ($M'c^2 + Q$). That is

$$M_0c^2 = M'c^2 + Q = E' \text{ (say)} + Q \quad \text{... (6)}$$

From Eqn. (5), $\quad p'c = Q$ ... (7)

From Eqn. (6) $\quad E' = M_0c^2 - Q$ ... (8)

For recoiling atom of rest mass M'_0,

$$\left(M'_0c^2\right)^2 = E'^2 - p'^2c^2$$
$$= (M_0c^2 - Q)^2 - (Q)^2 \text{ using Eqns. (7) \& (8).}$$

$$\therefore \quad \left(M'^2_0\, c^2\right)^2 = (M_0c^2 - Q + Q)\,(M_0c^2 - Q - Q)$$
$$= M_0c^2\,(M_0c^2 - 2Q) = (M_0c^2)^2 - 2M_0c^2Q \qquad \text{... (9)}$$

Now $\left(M'_0c^2\right)$ = Rest mass energy of the atom *after* emission of photon.

(M_0c^2) = Rest mass energy of the atom *before* emission of photon when the atom was in the excited state.

$\therefore$ $(M_0c^2) - (M'_0c^2) = Q_0$ (say) = Excitation energy of the atom or $M'_0c^2 = M_0c^2 - Q_0$.

Substituting this in Eqn. (9) we get $(M_0c^2 - Q_0)^2 = (M_0c^2)^2 - 2M_0c^2Q$

$$\therefore \quad M_0^2\,c^4 + Q_0^2 - 2M_0c^2Q_o = M_0^2\,c^4 - 2M_0c^2Q$$

$$\therefore \quad 2M_0c^2Q = 2M_0c^2Q_0 - Q_0^2$$

or
$$Q = Q_0\left(1 - \frac{Q_0}{2\,M_0\,c^2}\right) \qquad \text{... (10)}$$

That is, energy of the emitted photon Q is *less* than the excitation energy Q_0 of the atom. In other words only a part of the excitation energy is given to the photon, the remaining energy is retained as the kinetic energy of the recoiling atom.

7.3 PAIR PRODUCTION, PAIR ANNIHILATION

(A) Pair Production

Simultaneous production of a pair consisting of a particle and it's antiparticle (electron e^- and position e^+ pair for example) at the expense of the entire energy ($h\nu$) of a photon is called pair production. It is a very convincing example of conversion of electromagnetic energy into rest mass energy and kinetic energy of particles.

A particle and its antiparticle are always created together in a pair in order to conserve certain conservation laws which are of no interest to us here. We concern ourselves only with two conservation laws of mechanics namely that of mass-energy and linear momentum.

The minimum energy $h\nu_{\min}$ needed to create a particle—antiparticle pair is by conservation of energy given to be

$h\nu_{\min} = 2\,m_0c^2$ where m_0c^2 is the rest mass energy of each particle in the pair. Since the rest mass energy of an electron or a positron is 0.51 MeV, the minimum energy or the threshold photon energy for electron—positron pair production is 1.02 MeV.

When the photon energy $h\nu > 2\,m_0c^2$, the excess energy appears as the Kinetic energy T of the pair. By conservation of energy,

$$h\nu = (m_0c^2 + \mathrm{T}_+) + (m_0c^2 + T_-) = 2\,m_0c^2 + T_+ + T_- \text{ in an obvious notation.}$$

However conservation of linear momentum precludes a simple conversion of a photon into a particle-antiparticle pair. This is easy to see.

Suppose in some inertial frame s, a photon of momentum $h\nu/c$ disappears and a particle-antiparticle pair is created. Let us study this process in the c. of m. frame of the particle-antiparticle pair. In this frame, the final momentum is zero. Therefore the initial linear momentum is zero. This is impossible, since the photon speed is always c in every inertial frame. Linear momentum of a photon can never be zero. In other words, a photon cannot decay or disappear spontaneously into a particle-antiparticle pair in free space.

Pair production is possible only if a suitable particle such as a massive atomic nucleus can share some of the photon momentum.

In pair production a photon ($h\nu \geq 2m_0c^2$) loses all its energy and hence disappears in a close encounter with an atomic nucleus and simultaneously an electron-positron (particle-antiparticle) pair is produced.

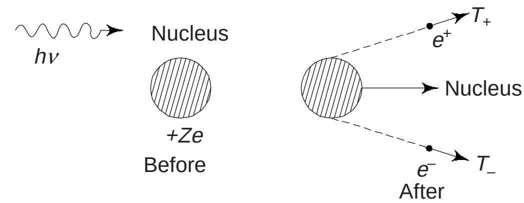

Fig. 7.5 Pair Production

For conservation of energy $h\nu = 2m_0c^2 + T_+ + T_-$ neglecting the extremely small (see below) kinetic energy of the recoiling nucleus. The positron is repelled by the nucleus whereas the electron is attracted by it. Hence the kinetic energy of the positron T_+ is slightly greater than T_-. We neglect this small difference and take $T_+ = T_-$. Then

$$h\nu = 2m_0c^2 + 2T_+ = 2(m_0c^2 + T_+) = 2E$$

where E is the total energy of electron or positron.

$$\therefore \qquad \frac{h\nu}{c} = \frac{2E}{c} = \frac{2}{c}\left(p^2c^2 + m_0^2c^4\right)^{1/2} \quad \text{where } p \text{ is the momentum of } e^+ \text{ or } e^-.$$

$$\therefore \qquad \frac{h\nu}{c} = 2(p^2 + m_0^2c^2)^{1/2}$$

$$\therefore \qquad \frac{h\nu}{c} > 2p$$

This equation shows that the photon momentum is greater than the total momentum ($2p$) of the pair even when they travel in forward direction. The conservation of linear momentum requires that the nucleus recoils to share the photon momentum.

A simple example will convince you that though the nucleus carries appreciable momentum, its kinetic energy is negligible.

Suppose a photon of energy $h\nu = 2m_0c^2 = 1.02$ MeV creates an electron-positron pair without giving any momentum to them. The entire momentum $h\nu/c$ is then given to the nucleus. This is equal to 1.02 MeV/c $= 1.02 \times 5.34 \times 10^{-22}$ kg m/s.

$$= 5.45 \times 10^{-22} \text{ kg m/s.}$$

Velocity given to the nucleus is small and using Newtonian mechanics its kinetic energy is $T = p^2/2$ m. Suppose the nucleus is that of lead for which mass is $m = 207$ u $= 207 \times 1.66 \times 10^{-27}$ kg.

$$\therefore \qquad T = \frac{(5.45\times10^{-22})^2}{2\times207\times1.66\times10^{-27}}\,\frac{1}{1.6\times10^{-19}} \text{ eV}$$

$$= 2.71 \text{ eV.}$$

This energy is negligible in comparison with the incident photon energy of 1.02×10^6 eV.

Electron-positron pair production is commonly observed in the interaction of high energy photons with matter.

Pair Annihilation

The annihilation of a particle-antiparticle pair and the concomitant creation of photons is the inverse of pair production. An electron and a positron which are essentially at rest near one another unite and are annihilated. Matter disappears, and radiant energy appears in its place. The total linear momentum of the two particles is initially zero. Therefore we cannot have only one photon created. A single photon cannot have zero momentum. Momentum can be conserved when two photons are created. Production of three or more photons is possible but probability of such processes turns out to be much less.

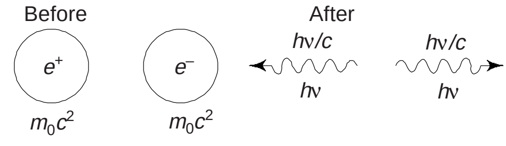

Fig. 7.6 Pair Annihilation

When two photons are created by pair annihilation of an electron and a positron at rest, for energy conservation we must have

$$2m_0c^2 = h\nu_1 + h\nu_2 \qquad \text{... (1)}$$

where m_0 = rest mass of electron = rest mass of positron.

For momentum conservation we must have

$O = \dfrac{h\nu_1}{c} - \dfrac{h\nu_2}{c}$ where ν_1 and ν_2 are the frequencies of the two photons created. Hence $\nu_1 = \nu_2$.

We can now write Eqn. (1) as

$$m_0c^2 = h\nu_{min}$$

$h\nu_{min}$ is the minimum energy of the outgoing photons and must equal m_0c^2, the rest mass energy of the electron or positron, 0.51 MeV.

When a positron passes through matter, it loses energy in successive collisions until it combines with an electron to form a bound system (similar to hydrogen atom) called positronium. Positronium promptly decays into photons.

7.4 ELASTIC COLLISIONS

(A) Compton Effect

A.H. Compton observed that when a monochromatic beam of *X*-rays (photons) is incident on a block of graphite, the scattered radiation consists of two components, one having the same wavelength as the incident radiation and the other having a slightly longer wavelength. Compton explained the presence of the longer wavelength radiation by considering the scattering as an elastic collision between a single photon and a free electron initially at rest. The collision is elastic in the sense that kinetic energy gained by the electron is equal to the energy lost by the photon and the electron is free in the sense that it is bound to the carbon parent rather loosely or with negligible binding energy—it can promptly leave the atom when it gains energy from the photon.

Let a photon of energy $h\nu$ and momentum $h\nu/c$, moving along the positive *X*-direction strike a stationary electron of rest mass m_0 as shown in Fig. 7.7 *Y*-axis is so chosen that the scattered photon and the recoiling electron are in the *X*-*Y* plane after the collision. The photon has energy $h\nu'$ and moves at an angle θ to the *X*-axis after the collision while the electron travels at speed v making an angle ϕ with the *X*-axis as shown.

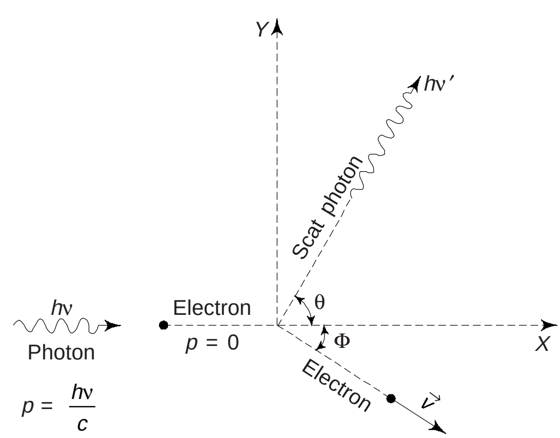

Fig. 7.7 Compton effect

By conservation of energy principle,

$$h\nu + m_0c^2 = h\nu' + mc^2$$

where $m = m_0/(1 - v^2/c^2)^{1/2}$. Since $\nu = \dfrac{c}{\lambda}$ and $\nu' = \dfrac{c}{\lambda'}$, above equation may be written as

$$\frac{hc}{\lambda} + m_0c^2 = \frac{hc}{\lambda'} + mc^2$$

or $$mc = \frac{h}{\lambda} + m_0c - \frac{h}{\lambda'}$$

$\therefore$ $$m^2c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} + m_0^2c^2 - \frac{2h^2}{\lambda\lambda'} + 2m_0c\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)$$

or $$\left(m^2 - m_0^2\right)c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'} + 2m_0c\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) \quad \text{... (1)}$$

Conservation of linear momentum along x and y directions gives respectively

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + mv\cos\Phi$$

$$O = \frac{h\nu'}{c}\sin\theta - mv\sin\Phi$$

Rearranging we get

$$mv\cos\Phi = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta$$

$$mv\sin\Phi = \frac{h}{\lambda'}\sin\theta$$

squaring and adding

$$m^2v^2 = \frac{h^2}{\lambda^2} - \frac{2h^2\cos\theta}{\lambda\lambda'} + \frac{h^2}{\lambda'^2} \quad \text{... (2)}$$

Since $$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}};\; m^2 - \frac{m^2v^2}{c^2} = m_0^2$$

$\therefore$ $$m^2v^2 = m^2c^2 - m_0^2c^2 = \left(m^2 - m_0^2\right)c^2$$

$\therefore$ R.H.S. of Eqn. (1) = R.H.S. of Eqn. (2)

$\therefore$ $$\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'} + 2m_0c\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) = \frac{h^2}{\lambda^2} - \frac{2h^2\cos\theta}{\lambda\lambda'} + \frac{h^2}{\lambda'^2}$$

$$\therefore \qquad 2m_0ch\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right) = \frac{2h^2}{\lambda\lambda'}(1-\cos\theta)$$

or increase in wavelength

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\theta) \qquad \text{... (3a)}$$

Or substituting the numerical values of h, m_0 and c,

$$\Delta\lambda = 0.02426\,(1-\cos\theta)\ \text{Å} \qquad \text{... (3b)}$$

This dependence of $\Delta\lambda$ on θ has been experimentally verified. The kinetic energies of the recoil electrons have also been found to be in agreement with their calculated values. The quantity h/m_0c = 0.02426 Å is known as Compton wavelength. All photons scattered through 90° have atleast this wavelength. It also turns out that the Compton wavelength is the wavelength of the annihilation radiation when an electron positron pair at rest annihilate to emit a pair of photons.

In Compton effect a stationary electron gains kinetic energy from an incident photon which suffers a loss of energy. On the other hand a fast moving electron can give part of its kinetic energy to a photon which is then scattered with increased energy that is reduced wavelength. This is inverse Compton effect.

(B) Symmetrical Elastic Collision of Identical Particles

We consider here collisions between two identical particles. The collision is elastic and rest masses of the colliding particles remain unaltered after the collision. Linear momentum as well as kinetic energy is conserved in such collisions. The collision is called symmetrical because after collision, the particles travel at equal angles (θ) to the direction of the incident particle. See Fig. 7.8.

Suppose a particle of rest mass m_0, momentum p_1 and total energy $E_1 = (m_0c^2 + T)$ is incident on an identical particle at rest.

In case of symmetrical elastic collision of identical particles, conservation of linear momentum along Y direction requires that both particle have the same magnitude of linear momentum and hence equal kinetic energy.

Conservation of linear momentum along X-axis gives

$$p_1 = 2p_2\cos\theta \qquad \text{... (1)}$$

Total energy of the system before collision is $(E_1 + m_0c^2)$ and after collision it is $2E_2$. See Fig. 7.8. Hence

$$E_1 + m_0c^2 = 2E_2 \qquad \text{... (2)}$$

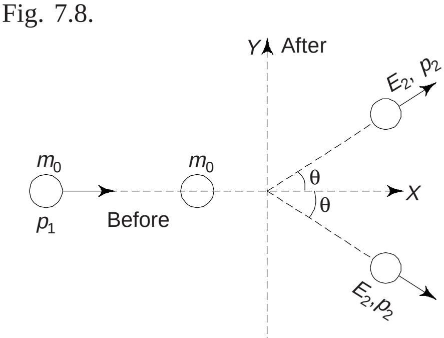

Fig. 7.8 Symmetrical Elastic Collision

From Eqn. (1) $\qquad p_1c = 2p_2c\cos\theta$

$$\therefore \qquad \sqrt{E_1^2 - m_0^2c^4} = 2\sqrt{E_2^2 - m_0^2c^4}\,\cos\theta$$

$$= 2\sqrt{\left(\frac{E_1 + m_0c^2}{2}\right)^2 - m_0^2c^4}\,\cos\theta$$

using Eqn. (2)

$$\therefore \qquad \cos\theta = \frac{\sqrt{E_1^2 - m_0^2c^4}}{\sqrt{(E_1 + m_0c^2)^2 - 4m_0^2c^4}}$$

$$= \frac{\sqrt{(E_1 + m_0 c^2)(E_1 - m_0 c^2)}}{\sqrt{(E_1 + 3m_0 c^2)(E_1 - m_0 c^2)}} = \frac{\sqrt{(E_1 + m_0 c^2)}}{\sqrt{(E_1 + 3m_0 c^2)}}$$

$\therefore$ $$\cos\theta = \sqrt{\frac{T + 2m_0 c^2}{T + 4m_0 c^2}} \qquad \text{... (3)}$$

because $E_1 = T + m_0c^2$

Note that when the kinetic energy of the incident particle is small ($T \ll m_0c^2$), $\cos\theta \to \frac{1}{\sqrt{2}}$ and $\theta \to 45°$ or $2\theta \to \frac{\pi}{2}$. This is in agreement with Newtonian mechanics according to which the angle between the two outgoing particles of equal mass is $\pi/2$ after the collision. On the other hand, when $T \gg m_0c^2$, $\cos\theta \to 1$ and $\theta \to 0$. This is an example of a general property of relativistic collisions. When the energy of the incident particle becomes much larger than the rest mass energy, products of collision tend to travel in the forward direction i.e. in the direction of the incident particle.

(C) Head-On Elastic Collision

Consider a head-on elastic collision in which a particle of rest mass m_0, energy e_0 and momentum p_0 collides head-on with a stationary mass M_0 which is knocked straight forward with energy E and momentum P. Let e and p denote respectively the energy and momentum of the projectile particle after the collision. See Fig. 7.9.

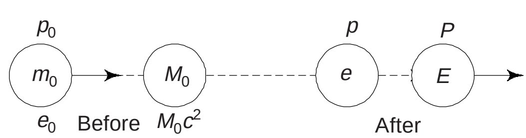

Fig. 7.9 Head-on Elastic Collision

As in elementary mechanics our aim is to find the momenta p and P in terms of the initial momentum and masses involved in the collision.

Conservation of momentum and energy give us respectively the equations:

$$p_0 = P + p \qquad \text{... (1)}$$

and $$e_0 + M_0c^2 = e + E \qquad \text{... (2)}$$

The most direct way of solving these equations is to write Eqn. (2) in terms of momenta p and P and then use Eqn. (1) to eliminate p and get the resulting equation for P.

Eqn. (2) may be written as

$$(e_0 + M_0c^2) - E = e \text{ or}$$

$$(e_0 + M_0c^2) - \sqrt{P^2c^2 + M_0^2c^4} = \sqrt{p^2c^2 - m_0^2c^4} = \sqrt{(p_0 - P)^2 c^2 + m_0^2 c^4}$$

Squaring both sides we get

$$(e_0 + M_0c^2)^2 + P^2c^2 + M_0^2c^4 - 2(e_0 + M_0c^2)\sqrt{P^2c^2 + M_0^2c^4} = p_0^2c^2 - 2p_0Pc^2 + P^2c^2 + m_0^2c^4$$

$$\therefore \quad \left(e_0^2 + M_0^2c^4 + 2e_0M_0c^2\right) + M_0^2c^4 - 2(e_0 + M_0c^2)\sqrt{P^2c^2 + M_0^2c^4} = p_0^2c^2 - 2p_0Pc^2 + m_0^2c^4$$

Using $e_0^2 = p_0^2c^2 + m_0^2c^4$ and simplifying we get,

$$M_0c^2(e_0 + M_0c^2) + p_0Pc^2 = (e_0 + M_0c^2)\sqrt{P^2c^2 + M_0^2c^4}$$

Squaring both sides and simplifying we get

$$Pc\left(e_0^2 + M_0^2c^4 + 2e_0M_0c^2 - p_0^2c^2\right) = 2M_0c^2(e_0 + M_0c^2)\,p_0c$$

Using $\quad e_0^2 - p_0^2c^2 = m_0^2c^4$

$$Pc = \frac{2p_0cM_0c^2(e_0 + M_0c^2)}{\left(m_0^2c^4 + 2e_0M_0c^2 + M_0^2c^4\right)} \qquad \text{... (3)}$$

Substituting this in equation $p_0c = pc + Pc$ we get

$$pc = p_0c - Pc = p_0c - \frac{2p_0cM_0c^2(e_0 + M_0c^2)}{\left(m_0^2c^4 + 2e_0M_0c^2 + M_0^2c^4\right)}$$

$$= \frac{p_0cm_0^2c^4 + 2e_0p_0cM_0c^2 + p_0cM_0^2c^4 - 2p_0cM_0c^2(e_0 + M_0c^2)}{\left(m_0^2c^4 + 2e_0M_0c^2 + M_0^2c^4\right)}$$

$$= \frac{p_0c\left(m_0^2 - M_0^2\right)c^4}{\left(m_0^2c^4 + 2e_0M_0c^2 + M_0^2c^4\right)} \qquad \text{... (4)}$$

In the classical limit $e_0 = m_0c^2$ and it is easily seen that Eqns (3) and (4) reduce respectively to

$$P = \frac{2p_0M_0}{M_0 + m_0} \qquad \text{... (5a)}$$

and

$$p = \frac{p_0(m_0 - M_0)}{M_0 + m_0} \qquad \text{... (6a)}$$

In Newtonian mechanics, mass is considered constant and above equations are customarily written as

$$P = \frac{2p_0M}{M + m} \qquad \text{... (5b)}$$

and

$$p = \frac{p_0(m - M)}{m + M} \qquad \text{... (6b)}$$

If $m = M$; $P = p_0$ and $p = 0$. Thus the incident particle stops and the struck particle moves on with the entire momentum. This is a well known result.

We also note that when $M_0 = m_0$, P reduces to p_0 and p reduces to zero as seen from Eqns. (3) and (4) respectively.

(D) Elastic Collision of Equal Rest Mass Particles

Collisions of high energy protons with protons or neutrons is a rich source of information in nuclear physics. Very often it is a good approximation to consider protons and neutrons as particles of equal rest mass. Hence the importance of collisions between particles of equal rest mass.

Collision between two protons can be elastic or inelastic. In the first case called elastic scattering studied here, the two colliding protons merely bounce apart like billiard balls. Their rest masses remain unchanged.

A particle of rest mass m_0, total energy E_a and momentum $\vec{p}_a$ strikes a stationary particle of equal rest mass m_0. See Fig. 7.10. The incident particle is deflected through angle θ_1. It has momentum $\vec{p}_c$ and energy E_c. The struck particle has initial momentum $p_b = 0$ and energy $E_b = m_0c^2$. After the collision, it recoils with momentum $\vec{p}_d$ and energy E_d at angle θ_2.

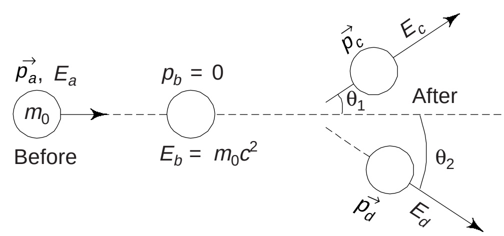

Fig. 7.10 Elastic collision of equal rest mass particles

The total momentum of the system is $p = p_a$ and its total energy is $E = (E_a + m_0c^2)$. Hence the velocity of the c of m of the system is given by

$$\beta = \frac{p_a c}{E_a + m_0 c^2}$$

In a frame moving with this velocity relative to the laboratory, the total linear momentum of the system is zero both before and after the collision.

In the c. of m. frame the two particles approach each other with equal and opposite momenta. After the collision, they are deflected through the same angle θ with equal and opposite momentum.
See Fig. 7.11

In the c. of m. frame, the energy of the system is given by

$$E'^2 = E^2 - p^2c^2 = (E_a + m_0c^2)^2 - p_a^2 c^2.$$

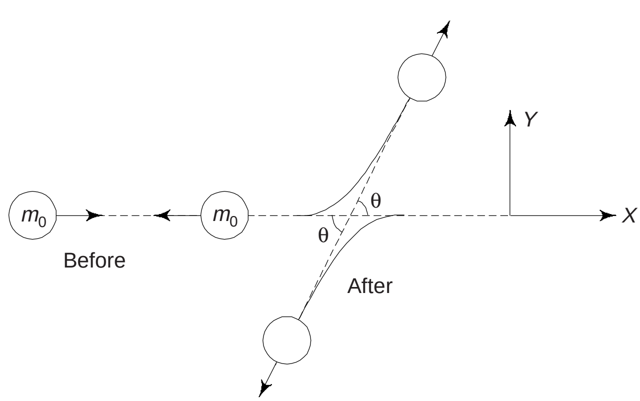

Fig. 7.11 Collision in c. of m. frame

$$= E_a^2 - p_a^2 c^2 + \left(2 m_0 E_a c^2 + m_0^2 c^4\right)$$

$$= m_0^2 c^4 + 2m_0 E_a c^2 + m_0^2 c^4$$

$$= 2m_0 c^2 (E_a + m_0 c^2) \quad \text{... (1)}$$

Note that in the c. of m. frame, we have treated the energy as if there were a single particle of that energy. It is worthwhile to find the momenta and energies of the individual particles by using the appropriate transformation equations.

In the c. of m. frame, the momentum of the incident particle is given by

$$p_a' c = \Gamma (p_a c - \beta E_a) \text{ where}$$

$$\beta = \frac{p_a c}{E_a + m_0 c^2}, \ \Gamma = \frac{1}{\sqrt{1-\beta^2}} \ \text{and} \ p_a c = \sqrt{E_a^2 - m_0^2 c^4}$$

$$\therefore \quad p_a' c = \frac{1}{\sqrt{1 - \dfrac{p_a^2 c^2}{(E_a + m_0 c^2)^2}}} \left(p_a c - \frac{p_a c}{E_a + m_0 c^2} \cdot E_a \right)$$

$$= \frac{E_a + m_0 c^2}{\sqrt{2 m_0 c^2 (E_a + m_0 c^2)}} \left(p_a c - \frac{p_a c E_a}{E_a + m_0 c^2} \right)$$

$$= \frac{E_a + m_0 c^2}{\sqrt{2 m_0 c^2 (E_a + m_0 c^2)}} \left(\frac{m_0 c^2}{E_a + m_0 c^2} \right) p_a c$$

$$= \sqrt{\frac{m_0 c^2}{2(E_a + m_0 c^2)}} \, p_a c = \sqrt{\frac{m_0 c^2}{2(E_a + m_0 c^2)}} \sqrt{E_a^2 - m_0^2 c^4}$$

$$= \sqrt{\frac{m_0 c^2 (E_a - m_0 c^2)}{2}} \quad \text{... (2)}$$

Similarly $\quad p_b' c = \Gamma(p_b c - \beta E_b) = \Gamma (0 - \beta m_0 c^2)$

$$= -\frac{E_a + m_0 c^2}{\sqrt{2 m_0 c^2 (E_a + m_0 c^2)}} \frac{p_a c \cdot m_0 c^2}{E_a + m_0 c^2} = -\sqrt{\frac{m_0 c^2}{2(E_a + m_0 c^2)}} \, p_a c = - p_a' c \quad \text{... (3)}$$

This verifies that the particles have equal and opposite momenta in the c. of m. frame.
Energy of the incident particle in the c. of m. frame is

$$E_a' = \Gamma (E_a - \beta p_a c)$$

Proceeding as above, it is easily shown that $E'_a = \dfrac{\sqrt{m_0 c^2 \,(E_a + m_0 c^2)}}{(2)^{1/2}}$ ... (4)

Similarly energy of the struck particle is given by

$$E'_b = \Gamma \,(E_b - \beta p_b c)$$

$$= \frac{E_a + m_0 c^2}{\sqrt{2 m_0 c^2 \,(E_a + m_0 c^2)}} \,(m_0 c^2 - 0) = \sqrt{\frac{m_0 c^2 \,(E_a + m_0 c^2)}{(2)}} = E'_a \quad \text{... (5)}$$

Also $E'_a + E'_b = \sqrt{2 m_0 c^2 \,(E_a + m_0 c^2)} = E'$ ... (6)

In the c. of m. system, the particles approach each other with equal and opposite momenta. Each of them is deflected by the same angle θ as shown. After collision, the two particles travel in opposite directions so that the total linear momentum is again zero. After the elastic collision, the momenta are altered only in direction and the energies remain unaltered. From Fig. 7.11, it is seen that

$$p'_{ax} c = \cos\,\theta \,\frac{\sqrt{m_0 c^2 \,(E_a - m_0 c^2)}}{\sqrt{2}} ;\; p'_{ay}\, c = \sin\theta \,\frac{\sqrt{m_0 c^2 \,(E_a - m_0 c^2)}}{\sqrt{2}}$$

$$p'_{bx} c = -\cos\,\theta \,\sqrt{\frac{m_0 c^2 \,(E_a - m_0 c^2)}{2}} ;\; p'_{by} c = -\sin\theta \,\frac{\sqrt{m_0 c^2 \,(E_a - m_0 c^2)}}{\sqrt{2}}$$

$$E'_a = E'_b = \frac{\sqrt{m_0 c^2 \,(E_a + m_0 c^2)}}{\sqrt{2}}$$

In the laboratory frame, the momentum of the incident particle is given by

$$p_{ax} c = \Gamma\left(p'_{ax} c + \beta E'_a\right)$$

Substituting for Γ, p'_{ax} etc, we get

$$p_{ax} c = \frac{\sqrt{E_a^2 - m_0^2 c^4}}{2} \,(1 + \cos\,\theta)$$

$$p_{ay} c = p'_{ay}\, c = \sin\theta \,\frac{\sqrt{m_0 c^2 \,(E_a - m_0 c^2)}}{\sqrt{2}}$$

$$\tan\,\theta_1 = \frac{p_{ay}}{p_{ax}} = \sqrt{\frac{2 m_0 c^2}{(E_a + m_0 c^2)}} \,\tan\frac{\theta}{2}$$

Similarly

$$\tan\,\theta_2 = \sqrt{\frac{2 m_0 c^2}{(E_a + m_0 c^2)}} \,\tan\left(\frac{\pi - \theta}{2}\right)$$

For low energy of incident particle, $E_a \approx m_0c^2$.

Then $$\theta_1 \approx \frac{\theta}{2} \text{ and } \theta_2 \approx \frac{\pi}{2} - \frac{\theta}{2} \text{ and } \theta_1 + \theta_2 = \pi/2.$$

This agrees with the well known classical result that in elastic collision between two particles of equal masses, after the collision, the particles go off at right angles to each other. As the incoming energy E_a increases, relativistic effect makes θ_1 and θ_2 smaller and smaller; that is the particles tend to move more and more in the forward direction.

7.5 INELASTIC COLLISIONS

In inelastic collisions between protons, kinetic energy of the incoming protons changes into rest mass energy of particles such as pions, mesons etc. Production of a proton anti-proton pair is one such example.

A. Production of Proton Antiproton Pair

As the kinetic energy of the incoming protons is increased more and more massive particles are produced in inelastic collisions between protons. As an illustration we shall calculate the minimum energy of the proton beam required to produce a proton antiproton pair. This is called the threshold energy.

Consider the inelastic collision of protons resulting in proton antiproton pair described by the reaction $p + p = p + p + p + \bar{p}$.

(In such collisions a proton and an antiproton are produced simultaneously as required by certain conservation laws such as the law of conservation of charge).

We shall study the collision in c. of m. frame

Fig. 7.12 Production of proton antiproton pair

In the c. of m. frame of reference, two protons approach each other with kinetic energies E_a' and $E_b' = E_a'$. The total linear momentum is zero. The total initial energy is $E' = E_a' + E_b'$.

In the c. of m. frame, the minimum energy required to produce a proton antiprotion pair is that which will produce all the four particles ($3p$ and $\bar{p}$) with zero kinetic energy, that is; all the four particles are at rest in the c. of m. frame. Hence the total energy of the system of particles in the c. of m. frame before or after the collision must be $4m_0c^2$ where m_0c^2 is the rest mass energy of a proton or an antiproton. Thus $E' = 4\ m_0c^2$.

Since $E_1^2 - p_1^2c^2 = E_2^2 - p_2^2c^2$ we must have

$E'^2 - (0)^2 = (E_a + m_0c^2)^2 - p^2c^2$ where E_a is the total energy and p the linear momentum of the incoming proton in the laboratory frame.

$$\therefore \quad E'^2 = \left(E_a^2 - p^2c^2\right) + 2E_am_0c^2 + m_0^2c^4 = 2m_0^2c^4 + 2E_am_0c^2$$

$$= 2m_0c^2\ (E_a + m_0c^2)$$

$$\therefore \quad (4m_0\ c^2)^2 = 2m_0c^2\ (E_a + m_0c^2)$$

$$\therefore \quad E_a = 7m_0c^2$$

This is the total energy of the incident proton. Of this, m_0c^2 is its rest mass energy. Hence the kinetic energy to which the incident proton beam must be accelerated for production of proton antiproton pair is $6m_0c^2 = 6(938 \text{ MeV}) = 5628 \text{ MeV}$.

7.6 ENERGY AVAILABLE FOR CONVERSION INTO REST MASS

In laboratory experiments particles such as protons may be accelerated to high energies for creation of mass such as pions or proton antiprotons etc. However because of conservation of momentum, product particles do possess linear momentum and therefore they also possess kinetic energy. Obviously all the kinetic energy of the incoming particles cannot be available for conversion into mass.

Suppose a particle of rest mass m_1, momentum p_1 and total energy E_i is incident on a stationary target particle of rest mass m_2. Let M denote the rest mass of one or more particles produced. M is the (total) rest mass of the system after the reaction. M is maximum when kinetic energy of the product particle or particles is zero in the c. of m. frame. In the c. of m. frame, the products produced are then stationary. As usual the total linear momentum is zero.

In the laboratory frame the total energy is $(E_i + m_2c^2)$ and momentum is p_i. In the c. of m. frame the total energy is Mc^2 and momentum is zero.

Using the invariance of $[E^2 - (Pc)^2]$, we get

$$(E_i + m_2c^2)^2 - p_i^2c^2 = (Mc^2)^2 - (o)^2$$

$$\therefore \quad \left(E_i^2 - p_i^2 c^2\right) + 2E_im_2c^2 + m_2^2c^4 = (Mc^2)^2$$

$$\therefore \quad m_1^2c^4 + 2E_im_2c^2 + m_2^2c^4 = (Mc^2)^2 \qquad \text{... (1)}$$

The total energy available in the c. of m. frame is Mc^2. It is given by $(m_1^2c^4 + 2E_im_2c^2 + m_2^2c^4)^{1/2}$.

It includes the sum of the rest mass energies $(m_1c^2 + m_2c^2)$ of the colliding particles.

In the extreme relativistic case, when $E_i \gg m_1c^2$ and $E_2 \gg m_2c^2$; the total energy available for conversion into mass is approximately equal to $(2m_2c^2E_i)^{1/2}$. Thus at extremely high energies, the energy available in the c. of m. frame of reference increases approximately only as the square root of E_i the total energy of the incident particle.

Suppose next that in the collision an extra rest mass Δm is created i.e. let

$$M = m_1 + m_2 + \Delta m$$

Then Eqn. (1) can be written as

$$(Mc^2)^2 = (m_1c^2 + m_2c^2 + \Delta mc^2)^2$$

$$= (m_1^2c^4 + m_2^2c^4 + 2E_im_2c^2)$$

$$\text{or} \quad (m_1c^2 + m_2c^2 + \Delta mc^2)^2 = m_1^2c^4 + m_2^2c^4 + 2m_2c^2(m_1c^2 + T) \qquad \text{... (2)}$$

where $E_i = m_1c^2 + T$; T being the kinetic energy of the incident particle.

On expanding the left hand side of Eqn. (2) and cancelling the terms common on either side, we get,

$$\Delta mc^2 [\Delta mc^2 + 2m_1c^2 + 2m_2c^2] = 2m_2c^2T$$

$$\text{or} \quad T = \Delta mc^2 \left[1 + \frac{m_1}{m_2} + \frac{\Delta m}{2m_2}\right] \qquad \text{... (3)}$$

This is the kinetic energy in the laboratory frame of reference which is required to produce one or more particles of total rest mass Δm.

As an example of the use of above equation, consider the reaction $p + p \rightarrow p + p + \pi^\circ$ for the production of neutral pions. Taking $m_{\pi^\circ} = 264\ \mathrm{m_e}$, $m_p = 1836\ \mathrm{m_e}$ and $m_e c^2 = 0.51$ MeV, we get for the threshold energy

$$T = m_{\pi^\circ} c^2 \left[1 + \frac{m_p}{m_p} + \frac{m_{\pi^\circ}}{2m_p}\right]$$

$$= 264 \times 0.51 \left[2 + \frac{264}{2 \times 1836}\right] \text{ MeV} = 279 \text{ MeV}$$

Again consider Eqn. (1) above. It can be written as

$$m_1^2 c^4 + m_2^2 c^4 + 2 E_i m_2 c^2 = (m_1 c^2 + m_2 c^2 + \Delta m c^2)^2$$

$$\therefore \qquad \Delta m c^2 = \left(m_1^2 c^4 + m_2^2 c^4 + 2 E_i m_2 c^2\right)^{1/2} - (m_1 c^2 + m_2 c^2)$$

This is the energy available to produce new mass. When $m_1 = m_2 = m_0$ say, above equation becomes

$$E_{\text{available}} = \Delta m c^2 = \left(2 m_0^2 c^4 + 2 E_i m_0 c^2\right)^{1/2} - (2 m_0 c^2) \qquad \text{... (4)}$$

Let $E_i = m_0 c^2 + T$ where T is the kinetic energy of the projectile particle.
In the non relativistic case, $T \ll m_0 c^2$

Then $$E_{\text{available}} = [2 m_0 c^2 (2 m_0 c^2 + T)]^{1/2} - 2 m_0 c^2 \qquad \text{... (5)}$$

$$= 2 m_0 c^2 \left(1 + \frac{T}{2 m_0 c^2}\right)^{1/2} - 2 m_0 c^2$$

$$\approx 2 m_0 c^2 \left(1 + \frac{1}{2} \frac{T}{2 m_0 c^2}\right) - 2 m_0 c^2 = \frac{T}{2}$$

In the super relativistic case $E_i \gg m_0 c^2$. We can now write Eqn. (4) as

$$E_{\text{available}} = [2 m_0 c^2 (m_0 c^2 + E_i)]^{1/2} - (2 m_0 c^2)$$

$$\approx (2 m_0 c^2 E_i)^{1/2} - 2 m_0 c^2$$

$$= 2 m_0 c^2 \left(\sqrt{\frac{E_i}{2 m_0 c^2}} - 1\right)$$

For a proton of rest mass energy 1 GeV and total energy $E_i = 300$ GeV say, the energy available for conversion into mass is

$$E_{\text{available}} = 2 \text{ GeV} \left(\sqrt{\frac{300}{2}} - 1\right) = 22.50 \text{ GeV only!}$$

Such huge losses of energy can be avoided by using colliding beams, that is collisions of two beams travelling in opposite directions.

7.7 ENERGETICS OF NUCLEAR REACTIONS (INELASTIC COLLISIONS)

Identity of a nucleus can be changed by bombarding it with a suitable high energy particle. Such a process is called a nuclear reaction. For instance when an alpha particle (helium nucleus $_2He^4$) strikes a nitrogen 14 nucleus, a proton and an oxygen 17 nucleus are produced. The nuclear reaction is

$$_7N^{14} + {_2He^4} \rightarrow {_8O^{17}} + {_1H^1}\ (p)$$

This reaction is often written in short as

$$_7N^{14}\ (\alpha, p)\ {_8O^{17}}$$

A general nuclear reaction may be written as $(X(x, y)Y$, where X is the target nucleus, x the bombarding particle, y the emergent light mass particle and Y the product nucleus. See Fig. 7.13.

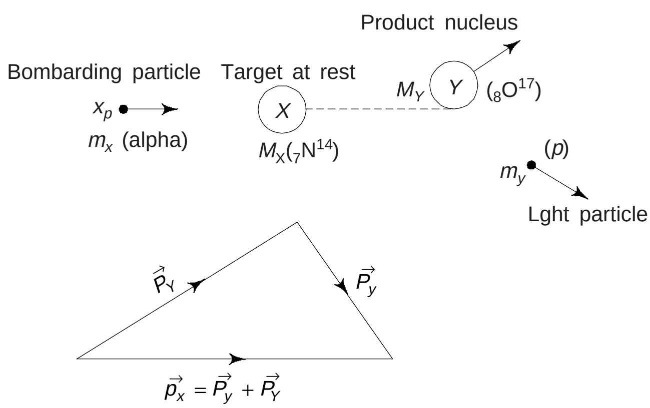

Fig. 7.13 Nuclear reaction

The target nucleus X is assumed to be at rest so that its kinetic energy $T_X = 0$. The kinetic energies of particles x, y and Y are denoted respectively by T_x, T_y and T_Y.

The Q of a nuclear reaction is defined as the total energy released in the reaction. That is, Q is defined as the kinetic energy coming out of the reaction minus the kinetic energy going into the reaction.

$$Q = (T_y + T_Y) - (T_x + T_X)$$

$$= (T_y + T_Y) - T_x\ (\because T_X = 0) \qquad \text{... (1)}$$

The total relativistic energy (inclusive of mass) going into the reaction is

$(m_x c^2 + T_x) + (M_X c^2 + T_X) = (m_x c^2 + T_x) + M_X c^2$

The total relativistic energy coming out of the reaction is $(m_y c^2 + T_y) + (M_Y c^2 + T_Y)$.

Hence from conservation of energy principle we see that

$$(m_x c^2 + T_x) + M_X c^2 = (m_y c^2 + T_y) + (M_Y c^2 + T_Y)$$

$$\therefore \quad Q = (T_y + T_Y) - T_x = (m_x + M_x)c^2 - (m_y + M_Y)c^2$$

= (Rest mass energy going into the reaction) – (Rest mass energy coming out of the reaction)

Thus Q can also be defined as rest mass energy going into the reaction minus the rest mass energy coming out of the reaction.

A nuclear reaction is called exoergic if $Q > 0$. In exoergic reactions kinetic energy coming out of the reaction exceeds the kinetic energy going into the reaction. This can happen only at the expense of rest mass. Thus in exoergic nuclear reactions part of the rest mass going into the reaction gets converted into kinetic energy of the particles emerging out of the reaction.

A nuclear reaction is endoergic if $Q < 0$. In endoergic reactions, total rest mass coming out of the reaction exceeds the total rest mass going into the reaction. This can happen only at the expense of kinetic energy. Thus in endoergic nuclear reactions, part of the kinetic energy of the bombarding particle (x) is converted into rest mass.

Since energy is released in exoergic reactions, in principle, an exoergic reaction can occur even when the kinetic energy of the bombarding particle is nearly zero. On the other hand, an endoergic reaction cannot occur unless the bombarding particle carries sufficient kinetic energy. The minimum kinetic energy of the bombarding particle required to make an endoergic reaction energetically possible would have been equal to the magnitude of Q provided all the kinetic energy of the bombarding particle could be converted into mass with emergent particles (y and Y) stationary as a result of the nuclear reaction. This is impossible because it violates the principle of conservation of linear momentum.

In the laboratory frame, the bombarding particle (x) carries linear momentum $(\vec{p}_x)$ as well as kinetic energy (T_x). Therefore the total momentum of the emerging particles $(\vec{p}_y + \vec{p}_Y)$ must be equal the momentum $(\vec{p}_x)$ of the incident particle. Emerging particles (y and Y) are in motion and carry kinetic energy. Hence only a part of incoming kinetic energy (T_x) is converted into mass. Obviously the kinetic energy of the incident particle must exceed $|Q|$ in order to initiate an endoergic reaction. The minimum kinetic energy of the incident particle for which an endoergic reaction is possible is called its threshold energy E_{th}. Thus $E_{th} > |Q|$.

Figure 7.14(a) and (b) show the nuclear reaction in the laboratory frame and the c. of m. frame respectively. Suppose the incident particle (x) has the minimum kinetic energy E_{th} required to make the reaction possible.

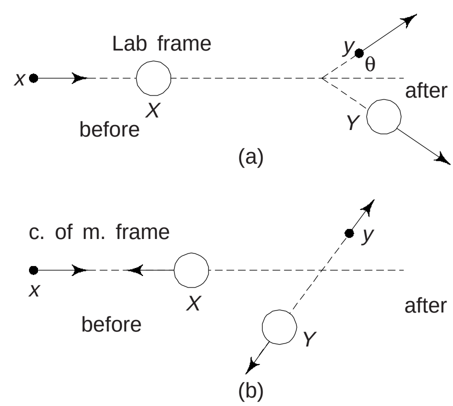

Fig. 7.14 (a) and (b) Energetics of nuclear reaction

In the laboratory frame the total energy and linear momentum of the system are given by

$$E_{\text{lab}} = (m_x c^2 + E_{th}) + M_X c^2 \quad \text{... (1)}$$

$$P_{\text{lab}} = p_x \quad \text{... (2)}$$

When the kinetic energy brought by the incident particle is equal to the threshold energy, reaction products are observed to be stationary in the c. of m. frame. Hence in the c. of m. frame, the total energy and linear momentum in the system are given respectively by

$$E_{\text{c.m.}} = (m_y + M_Y)c^2 \quad \text{... (3)}$$

$$P_{\text{c.m.}} = 0 \quad \text{... (4)}$$

Equating $E_{\text{lab}}^2 - p_{\text{lab}}^2 c^2 = E_{\text{c.m}}^2 - p_{\text{c.m}}^2 c^2$ we get

$$[(m_x c^2 + E_{th}) + M_X c^2]^2 - p_x^2 c^2 = (m_y c^2 + M_Y c^2)^2 \quad \text{... (5)}$$

The left hand side of this equation can be written as

$$\begin{aligned}(m_x c^2 + E_{th})^2 + (M_X c^2)^2 + 2M_X c^2 (m_x c^2 + E_{th}) - p_x^2 c^2 \\ &= [(m_x c^2 + E_{th})^2 - p_x^2 c^2] + M_X c^2 [M_X c^2 + 2(m_x c^2 + E_{th})] \\ &= E_x^2 - p_x^2 c^2 + M_X c^2 [M_X c^2 + 2(m_x c^2 + E_{th})] \\ &= (m_x c^2)^2 + M_X c^2 [M_X c^2 + 2(m_x c^2 + E_{th})]\end{aligned}$$

Hence Eqn. (5) can be written as

$$M_X c^2 \lfloor M_X c^2 + 2m_x c^2 + 2E_{th} \rfloor = (m_y c^2 + M_Y c^2)^2 - (m_x c^2)^2$$

$$\therefore \quad M_X c^2 \lfloor M_X c^2 + 2E_{th} \rfloor = (m_y c^2 + M_Y c^2)^2 - \lfloor (m_x c^2)^2 + 2m_x M_X c^4 \rfloor$$

$$\therefore \qquad (M_X c^2 + 2E_{th}) = \frac{(m_y c^2 + M_Y c^2)^2 - \lfloor (m_x c^2)^2 + M_X c^2\,(2 m_x c^2) \rfloor}{M_X c^2}$$

$$\therefore \qquad E_{th} = \frac{(m_y c^2 + M_Y c^2)^2 - \lfloor (m_x c^2)^2 + M_X c^2\,(2 m_x c^2 + M_X c^2) \rfloor}{2 M_X c^2}$$

$$= \frac{(m_y c^2 + M_Y c^2)^2 - (m_x c^2 + M_X c^2)^2}{2 M_X c^2}$$

$$= \frac{(m_y c^2 + M_Y c^2 + m_x c^2 + M_X c^2)(m_y c^2 + M_Y c^2 - m_x c^2 - M_X c^2)}{2 M_X c^2}$$

$$\therefore \qquad E_{th} = -\frac{Q(m_x + M_X + m_y + M_Y)}{2 M_X} \qquad \text{... (6)}$$

Note that $Q = (m_x c^2 + M_X c^2) - (m_y c^2 + M_Y c^2)$ is intrinsically a negative quantity for an endoergic reaction. Thus E_{th} is essentially a positive quantity as it should be. Above equation may also be written as

$$E_{th} = \frac{|Q|(m_x + M_x + m_y + M_Y)}{2 M_X} \qquad \text{... (7)}$$

Note that $(m_x + M_X + m_y + M_Y)$ is the total rest mass of all particles going into the reaction and coming out of the reaction. Above equation holds good even when there are more than two products of the reaction. Equation (7) can also be written as

$$E_{th} = \frac{|Q|\,(\text{Sum of rest masses of all particles entering and leaving the reaction})}{2\,(\text{Rest mass of the target particle})}$$

When $|Q|$ of an endoergic reaction is very small, very little new mass is created and $(m_x + M_X) \approx (m_y + M_Y)$. Eqn. (7) may now be written as

$$E_{th} \approx \frac{|Q|(m_x + M_X)}{M_X}$$

SUMMARY

When a unstable particle of rest mass M decays into two particles of rest mass M_1 and M_2 the two particles emitted must carry equal and opposite momenta if the unstable particle is stationary. The total energies carried by the decay products are then given by

$$E_1 = \frac{\left(M^2 + M_1^2 - M_2^2\right)c^2}{2M} \quad \text{and} \quad E_2 = \frac{\left(M^2 + M_2^2 - M_1^2\right)c^2}{2M}$$

In this decay, a part of the original rest mass energy (Mc^2) is converted into kinetic energy of the decay products.

When an atom (rest mass M_0) absorbs a photon of energy Q, in order to conserve linear momentum, the atom recoils in the direction of the incident photon with a velocity $v = \dfrac{Q}{c(M_0 + Q/c^2)}$.

When an atom (rest mass M_0) in an excited state emits a photon, it recoils with a momentum equal and opposite to that of the emitted photon. On account of this, the atom possesses kinetic energy. Hence the energy of the emitted photon Q is less than the excitation energy Q_0 of the atom.

$$Q = Q_0 \left(1 - \frac{Q_0}{2 M_0 c^2}\right)$$

Simultaneous production of a pair consisting of a particle and its antiparticle (electron and positron) at the expense of the entire energy ($h\nu$) of a photon is called a pair production.

Because of conservation of linear momentum, a photon cannot disappear spontaneously creating a particle and its antiparticle in free space. In pair production a photon (of energy $h\nu \geq 2m_0c^2$) losses all its energy and disappears in a close encounter with an atomic nucleus and simultaneously an electron positron pair is produced. The atomic nucleus carries substantial amount of the original momentum though it takes up negligible kinetic energy.

Pair annihilation is the inverse of pair production. When an electron-positron pair essentially at rest annihilate each other, two photons going in opposite directions are produced. Their energies are equal.

Compton collision is essentially an elastic collision between a photon and an electron. Part of the incident photon energy is given to the electron as its kinetic energy. The photon is therefore scattered with a reduced energy or increased wavelength. The increase in wavelength of a photon scattered through an angle θ is

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

In case of symmetrical elastic collision of identical particles (rest mass m_0) the angle 2θ between the two outgoing particles is obtained from

$\cos\theta = \sqrt{\dfrac{T + 2m_0c^2}{T + 4m_0c^2}}$ where T is the kinetic energy of the incident particle. As T increases $\theta \to 0$ i.e. products of collision tend to travel in the forward direction — a relativistic effect. However when $T \ll m_0c^2$, $\cos\theta \to \dfrac{1}{\sqrt{2}}$ and $2\theta \to \dfrac{\pi}{2}$. This is in agreement with Newtonian mechanics.

For head-on elastic collision of a particle of rest mass m_0, momentum p_0 and energy e_0 with a stationary particle of rest mass M_0, the momenta after the collision are given by

$$pc = \frac{p_0 c\left(m_0^2 - M_0^2\right)c^4}{\left(m_0^2 c^4 + 2e_0 M_0 c^2 + M_0^2 c^4\right)}$$

and

$$Pc = \frac{2p_0 cM_0 c^2 (e_0 + M_0 c^2)}{\left(m_0^2 c^4 + 2e_0 M_0 c^2 + M_0^2 c^4\right)}$$

In the classical limit ($e_0 = m_0c^2$), they reduce to

$p = \dfrac{p_0(m - M)}{m + M}$; $P = \dfrac{2pM}{M + m}$ in agreement with Newtonian mechanics.

In case of elastic collision between equal rest mass particles, relativistic mechanics gives results which in the limit of low energy agree with the well known classical result that the two particles of equal mass go off at right angles to each other after the collision.

In inelastic collisions, kinetic energy of the incident particle is converted into mass; that is new particles are created as for example $p + p \rightarrow p + p + p + \bar{p}$. In this collision a proton-antiproton pair is created at the expense of the kinetic energy of the incident proton. Minimum kinetic energy required of the incident proton is 6 m_0c^2 where m_0c^2 = 938 MeV = rest mass energy of proton/antiproton. This exceeds the rest mass energy of the pair (2 m_0c^2) because products of reaction cannot be stationary but must possess linear momentum (equal to initial linear momentum) and hence kinetic energy.

Evidently all the incident kinetic energy is not available for conversion into mass. At extremely high energies, the energy available for conversion into mass varies approximately as the square root of the total energy of the incident particle.

When a particle of rest mass m_1 is incident on a particle of rest mass m_2, the kinetic energy needed to produce an extra mass Δm is given by

$$T = \Delta mc^2 \left[1 + \frac{m_1}{m_2} + \frac{\Delta m}{2\, m_2}\right]$$

When $m_1 = m_2 = m_0$ say, the energy available for conversion into mass is

$$E_{\text{available}} = [2m_0c^2\ (2m_0c^2 + T)]^{1/2} - 2m_0c^2$$

In the non-relativistic case $T \ll m_0c^2$

$$E_{\text{available}} \approx \frac{T}{2} = \text{half the incident K.E.}$$

In the super-relativistic case when the total energy of the incident particle $E_i = \gg m_0c^2$,

$$E_{\text{available}} \approx 2m_0c^2 \left(\sqrt{\frac{E_i}{2\, m_0\, c^2}} - 1\right)$$

Identity of a nucleus can be changed by bombarding it with a suitable high energy particle. Such a process is called a nuclear reaction. A nuclear reaction may be written as $X(x, y)Y$, where X is the target nucleus (at rest), x is the bombarding particle, y the emergent light mass particle and Y the product nucleus.

The Q of a reaction is defined as the total energy released in the reaction.

Q = Kinetic energy coming out of the reaction minus the kinetic energy going into the reaction.

$Q = (T_y + T_Y) - (T_X + T_x) = (T_y + T_Y) - T_x$ when $T_X = 0$.

Q = Rest mass energy going into the reaction minus the rest mass energy coming out of the reaction.

$= (m_x + M_X)c^2 - (m_y + M_Y)c^2$

A nuclear reaction is called exoergic if $Q > 0$ and endoergic if $Q < 0$.

In endoergic reactions, mass is created at the expense of incident energy of the bombarding particle. The minimum kinetic energy of the incident particle for which an endoergic reaction is possible is called its threshold energy E_{th}. Conservation of linear momentum makes $E_{th} > |Q|$.

$$E_{th} = \frac{|Q|\,(m_x + M_X + m_y + M_Y)}{2\, M_X}$$

$$E_{th} = \frac{|Q|\,(\text{sum of rest masses of all particles entering and leaving the reaction})}{2(\text{rest mass of the target nucleus})}$$

ILLUSTRATIVE EXAMPLES

Example 1 A pion (Π^+) of rest mass 139.6 MeV/c^2 decays at rest into a muon (μ^+) (rest mass 105.7 MeV/c^2) and a zero rest mass particle called neutrino. Calculate the kinetic energy and the momentum of (a) the muon (b) the neutrino. If the pion was moving with a velocity of 0.6 c relative to the laboratory when it decayed, what would be the maximum and the minimum possible values of the kinetic energy of the muon as measured in the laboratory?

Solution

(a) For muon of rest mass M_1;

$$E_1' = \frac{\left(M^2 + M_1^2 - M_2^2\right)c^2}{2M} = \frac{(139.6)^2 + (105.7)^2 - (0)^2}{2(139.6)} = 109.8 \text{ MeV.}$$

Kinetic energy of muon $T_1' = E_1' - M_1c^2$

$= (109.8 - 105.7) = 4.1$ MeV.

(b) In case of neutrino;

$$E_2' = \frac{\left(M^2 + M_2^2 - M_1^2\right)c^2}{2M} = \frac{(139.6)^2 + (0)^2 - (105.7)^2}{2(139.6)} = 29.8 \text{ MeV.}$$

Kinetic energy of the neutrino $T'_2 = E'_2$ because its rest mass is zero.

Momentum of muon = momentum of neutrino = E'_2/c = 29.8 MeV/c.

Or for either particle $p'c$ = 29.8 MeV.

If the pion is in motion when it decays, the maximum and the minimum possible values of the kinetic energy of the muon are obtained when the muon is emitted in the forward and the backward directions respectively.

In the present case $\Gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \dfrac{1}{\sqrt{1-(0.6)^2}} = 1.25$

$\therefore$

$$(E_1)_{max} = \Gamma(E'_1 + vp') = 1.25(109.6 + 0.6 \times 29.8)$$
$$= 159.6 \text{ MeV.}$$
$$(T_1)_{max} = (E_1)_{max} - M_1c^2 = (159.6 - 105.7) = 53.9 \text{ MeV.}$$
$$(E_1)_{min} = \Gamma(E'_1 - vp') = 1.25\ (109.6 - 0.6 \times 29.8)$$
$$= 123.9 \text{ MeV.}$$
$$(T_1)_{min} = (E_1)_{min} - M_1c^2 = (123.9 - 105.7) = 18.2 \text{ MeV.}$$

Example 2 An unstable particle of rest mass M and momentum p decays in flight into two particles of rest masses M_1 and M_2, momenta p_1 and p_2 and total energies E_1 and E_2. Show that $M^2c^4 = (M_1 + M_2)^2c^4 + 2E_1E_2 - 2M_1M_2c^4 - 2p_1p_2c^2 \cos\theta$ where θ is the angle between the two product particles.

Solution Referring to Fig. 7.15, equations for conservation of energy and linear momentum are,

$$E = E_1 + E_2 = \sqrt{p_1^2c^2 + M_1^2c^4} + \sqrt{p_2^2c^2 + M_2^2c^4} \quad \text{... (1)}$$

$$p = p_1 \cos\alpha + p_2 \cos\beta \quad \text{... (2)}$$

$$0 = p_1 \sin\alpha - p_2 \sin\beta \quad \text{... (3)}$$

Squaring Eqns. (2) and (3);

$$p^2 = p_1^2\cos^2\alpha + p_2^2\cos^2\beta + 2p_1p_2\cos\alpha\cdot\cos\beta$$

$$0 = p_1^2\sin^2\alpha + p_2^2\sin^2\beta - 2p_1p_2\sin\alpha\cdot\sin\beta$$

Adding above two equations, we get

$$p^2 = p_1^2 + p_2^2 + 2p_1p_2(\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta)$$

$$\therefore \quad p^2 = p_1^2 + p_2^2 + 2p_1p_2\cdot\cos\theta \text{ (where } \theta = \alpha + \beta) \quad (4)$$

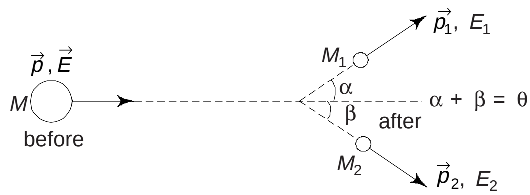

Fig. 7.15 For Illustrative Example 2

From Eqn. (1); $$E^2 = p^2c^2 + M^2c^4 = E_1^2 + E_2^2 + 2E_1E_2$$

$$\therefore \quad M^2c^4 = E_1^2 + E_2^2 + 2E_1E_2 - c^2(p_1^2 + p_2^2 + 2p_1p_2\cdot\cos\theta)$$

$$= \left(M_1^2c^4 + p_1^2c^2\right) + \left(M_2^2c^4 + p_2^2c^2\right) + 2E_1E_2 - c^2\left(p_1^2 + p_2^2 + 2p_1p_2\cdot\cos\theta\right)$$

$$= M_1^2c^4 + M_2^2c^4 + 2E_1E_2 - 2p_1p_2c^2\cdot\cos\theta$$

$$= (M_1c^2 + M_2c^2)^2 - 2M_1M_2c^4 + 2E_1E_2 - 2p_1p_2c^2\cdot\cos\theta$$

Example 3 A rocket starting from rest propels itself along a straight line through empty space by emitting photons and reaches a final velocity v relative to its initial rest frame. Show that

$$\frac{\text{initial mass of rocket}}{\text{final mass of rocket}} = \left(\frac{c+v}{c-v}\right)^{1/2}$$

Solution

Initial system consists of a rocket of mass say M_i at rest having energy M_ic^2 and momentum zero. Final system consists of a total number say N of photons, each having an energy E, momentum E/c in the backward direction and the rocket of rest mass M_f travelling with a velocity v in the forward direction.

By conservation of linear momentum,

$$0 = \Gamma M_f v - NE/c \text{ where } \Gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \quad NE = \Gamma M_f vc \quad \text{... (1)}$$

By conservation of energy;

$$M_ic^2 = \Gamma M_f c^2 + NE = \Gamma M_f c^2 + \Gamma M_f vc$$

$$= \Gamma M_f c(c + v)$$

$$\therefore \quad \frac{M_i}{M_f} = \frac{\Gamma}{c}(c+v) = \frac{c+v}{c\left(1-\frac{v^2}{c^2}\right)^{1/2}} = \sqrt{\frac{c+v}{c-v}}$$

Example 4 A neutral pion (m_0c^2 = 135 MeV) decays into two photons while travelling along positive X direction with a kinetic energy twice its rest mass energy. Find the momentum and energy of each of the two photons if one of the photons is observed to travel (a) along positive X direction (b) along positive Y direction.

Solution

(a) For the pion, the total energy is $3E_0$ where E_0 = 135 MeV = its rest mass energy.

Since $p^2c^2 = E^2 - E_0^2$, we have $pc = 2\sqrt{2}\,E_0$.

From conservation of energy and momentum, $E_1 + E_2 = 3E_0$ and $E_1 - E_2 = pc = 2\sqrt{2}\,E_0$. See figure.

Solving these two simultaneous equations, we get

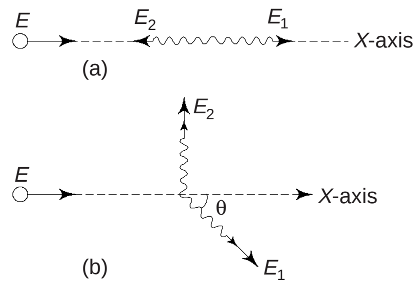

Fig. 7.16 For Illustrative Example 4

$$E_1 = \frac{\left(3+2\sqrt{2}\right)E_0}{2} = 393.4 \text{ MeV and } E_2 = \frac{\left(3-2\sqrt{2}\right)E_0}{2}$$

= 11.6 MeV. Their momenta are 393.4 MeV/c and 11.6 MeV/c respectively.

(b) Referring to the figure, conservation of energy and momentum give

$$E_1 + E_2 = 3E_0$$

$$E_1 \cos\theta = pc = 2\sqrt{2}E_0 \text{ and } E_1 \sin\theta = E_2$$

$\therefore$ $$E_1(1+\sin\theta) = 3E_0$$

Also $$E_1^2\cos^2\theta = 8E_0^2$$

and $E_1^2 + E_1^2\sin^2\theta + 2E_1^2\sin\theta = 9E_0^2.$

Adding the last two equations we get

$$2\,E_1^2(1+\sin\theta) = 17E_0^2$$

Also $$E_1(1+\sin\theta) = 3E_0^2$$

$\therefore$ $$2E_1 = \frac{17}{3}E_0$$

Hence $$E_1 = 382.3 \text{ MeV}$$

$\therefore$ $$E_2 = 22.7 \text{ MeV.}$$

Also it is easily shown that $\sin\theta = \dfrac{1}{17}$ or $\theta = 3.4°$.

Momenta of the photons are 382.3 MeV/c and 22.7 MeV/c.

Example 5 An atom in an excited state of energy E_0 above the ground state is observed to travel with a speed v. The atom decays to its ground state by emitting a photon of energy E. The act of photon emission brings the atom completely to rest. If the rest mass of the atom (ground state) is m, show that

$$E = E_0\left[1+\frac{E_0}{2mc^2}\right]$$

Solution

Let m' = rest mass of the atom in the excited state.

Then $$m'c^2 = mc^2 + E_0 \qquad \text{... (1)}$$

By conservation of energy principle

$$\Gamma m'c^2 = mc^2 + E \qquad \text{... (2)}$$

$$\Gamma = (1 - v^2/c^2)^{-1/2}$$

By conservation of linear momentum,

$$\Gamma m'v = E/c \qquad \text{... (3)}$$

Dividing Eqn. (2) by Eqn. (1);

$$\Gamma = \frac{mc^2 + E}{mc^2 + E_0} \quad \text{... (4)}$$

Similarly from Eqns. (2) and (3) we get

$$\frac{v}{c^2} = \frac{E/c}{mc^2 + E} \quad \text{or} \quad \frac{v}{c} = \frac{E}{mc^2 + E} \quad \text{... (5)}$$

$$\therefore \quad \frac{1}{\Gamma^2} = 1 - \frac{v^2}{c^2} = 1 - \frac{E^2}{(mc^2 + E)^2} \quad \text{... (6)}$$

From Eqns. (4) and (6), we find that

$$\frac{(mc^2 + E_0)^2}{(mc^2 + E)^2} = 1 - \frac{E^2}{(mc^2 + E)^2}$$

$$\therefore \quad (mc^2 + E_0)^2 = (mc^2 + E)^2 - E^2 = (mc^2 + 2E)mc^2$$

$$\therefore \quad m^2c^4 + E_0^2 + 2mc^2E_0 = m^2c^4 + 2mc^2E$$

$$\therefore \quad E = E_0 \left[1 + \frac{E_0}{2mc^2}\right]$$

Example 6 A photon of energy E collides with a stationary excited atom. As a result of the collision, the photon has its energy unchanged but its direction is reversed. If the atom is left in its ground state after the collision, find its initial excitation energy.

Solution Let Q denote the excitation energy. Then the rest mass of the excited atom is

$$m' = m + Q/c^2 \quad \text{... (1)}$$

where m is the rest mass of the atom in its ground state.

Let v denote the speed of the atom after the collision. Then by conservation of energy;

$$E + m'c^2 = E + \Gamma mc^2 \quad \text{... (2)}$$

where $\Gamma = (1 - v^2/c^2)^{-1/2}$

By conservation of linear momentum,

$$\frac{E}{c} = \Gamma mv - \frac{E}{c} \quad \text{... (3)}$$

From Eqn. (2); $\Gamma = \frac{m'}{m} = \frac{m + Q/c^2}{m}$ using (l).

$$\therefore \quad \Gamma = 1 + \frac{Q}{mc^2} \quad \text{or} \quad Q = mc^2 (\Gamma - 1) \quad \text{... (4)}$$

By Eqn. (3) $v = \frac{2E}{\Gamma mc}$

$$\therefore \quad \Gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} = \frac{1}{1 - \frac{4E^2}{\Gamma^2 m^2 c^4}} = \frac{\Gamma^2 m^2 c^4}{\Gamma^2 m^2 c^4 - 4E^2}$$

∴ $$m^2c^4 = \Gamma^2 m^2 c^4 - 4E^2$$

or $$\Gamma^2 = \frac{m^2c^4 + 4E^2}{m^2c^4}$$

∴ $$\Gamma = \left[1 + \left(\frac{2E}{mc^2}\right)^2\right]^{\frac{1}{2}}$$

Using this in Eqn. (4) we get

$$Q = mc^2\left[\left(1 + \left(\frac{2E}{mc^2}\right)^2\right)^{1/2} - 1\right]$$

Example 7 Show that the following processes are dynamically impossible. (a) A single photon strikes a stationary electron and gives up all its energy to it. (b) A single photon in an empty space is transformed into an electron and a positron. (c) A fast positron and a stationary electron annihilate producing only one photon.

Solution

(a) Suppose such a process is possible. Then conservation of momentum requires that $\frac{E}{c} + 0 = p$ or $E = pc$ where E is the energy of the photon and p is the electron momentum after photon absorption. Conservation of energy requires that

$$E + m_0c^2 = \sqrt{p^2c^2 + m_0^2c^4}$$

∴ $$pc + m_0c^2 = \sqrt{p^2c^2 + m_0^2c^4}$$

∴ $p^2c^2 + m_0^2c^4 + 2pcm_0c^2 = p^2c^2 + m_0^2c^4$ which is absurd.

(b) If the process is possible, then in an obvious notation conservation of momentum and energy give respectively

$$\frac{E}{c} = p_+ + p_- \quad \text{or} \quad E = p_+c + p_-c \qquad \text{... (1)}$$

and $$E = E_+ + E_- = \left(p_+^2c^2 + m_0^2c^4\right)^{1/2} + \left(p_-^2c^2 + m_0^2c^4\right)^{1/2} \qquad \text{... (2)}$$

Equations (1) and (2) are mutually inconsistent unless $m_0c^2 = 0$

(c) Consider the process is the c. of m. frame of the positron and electron. In this frame, initial momentum is zero. Hence final momentum must be zero. But there is no frame of reference in which the photon produced will have zero momentum. Obviously the process cannot take place in the c. of m. frame. Hence it cannot take place in any other inertial frame.

Example 8 A particle of rest mass m and kinetic energy T is incident on a stationary particle of rest mass M. As a result of the collision the incident particle is scattered through 90° and has a momentum P.

Show that after the collision the target particle has a momentum $(p^2 + 2Tm + T^2/c^2)^{1/2}$ and travels in a direction that makes angle $\theta = \tan^{-1}\left(\frac{p}{(2Tm + T^2/c^2)^{1/2}}\right)$ with the direction of the incident particle.

Solution Let p be initial momentum of the incident particle and p' that of the target particle after the collision. See Fig. 7.17. From conservation of linear momentum we must have

$$P = p' \sin\theta$$

$$p = p' \cos\theta$$

$$\therefore \quad \text{Tan}\theta = P/p \qquad \ldots (1)$$

Also $mc^2 + T$ = total energy of the incident particle

$$= \sqrt{p^2c^2 + m^2c^4}$$

$$m^2c^4 + 2\,T\,mc^2 + T^2 = p^2c^2 + m^2c^4$$

or $$p = \sqrt{2mT + T^2/c^2}$$

$$\therefore \quad \theta = \tan^{-1}\left[\frac{P}{\sqrt{2mT + T^2/c^2}}\right] \text{ as required.}$$

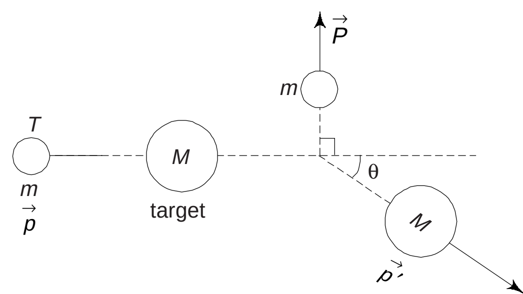

Fig. 7.17 For Illustrative Example 8

Also $$P^2 = p'^2 \sin^2\theta$$

$$p^2 = p'^2 \cos^2\theta$$

$$\therefore \quad p'^2 = P^2 + p^2 = P^2 + 2mT + T^2/c^2$$

$$\therefore \quad p' = \sqrt{P^2 + 2mT + T^2/c^2}$$

Example 9 A neutral pion (rest mass 135 MeV) having a kinetic energy of 1 GeV decays in flight into two photons. (a) Find their energies if they are emitted in opposite directions along the pion's original line of travel. (b) What is the angle between the two photons if they are emitted symmetrically with respect to the direction of pion travel?

Solution

(a) Refer to Fig. 7.18 From conservation of energy we see that

$$E_1 + E_2 = E = 135 + 1000 = 1135 \text{ MeV} \qquad \ldots (1)$$

From momentum conservation

$$\frac{E_1}{c} - \frac{E_2}{c} = p \quad \text{or} \quad E_1 - E_2 = pc = \sqrt{E^2 - m_o^2c^4}$$

$$\therefore \quad E_1 - E_2 = \sqrt{(1135)^2 - (135)^2} = 1127 \text{ MeV} \qquad \ldots (2)$$

Addition and subtraction of eqns. (1) and (2) gives

$$E_1 = 1131 \text{ MeV}; \; E_2 = 4 \text{ MeV}.$$

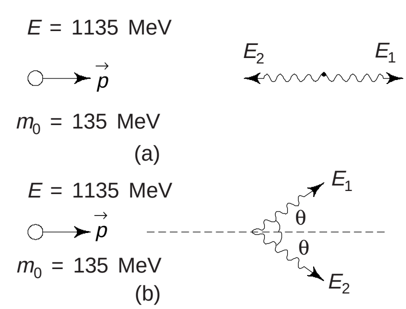

Fig. 7.18 For Illustrative Example 9

(b) From symmetry, we see that $E_1 = E_2$ because otherwise conservation of momentum is impossible. Also $E_1 + E_2 = 1135$ MeV.

$$\therefore \quad E_1 = E_2 = \frac{1135}{2} \text{ MeV.}$$

From momentum conservation, $\frac{E_1}{c}\cos\theta + \frac{E_2}{c}\cos\theta = p$ or $2E_1\cos\theta = pc$ or $\cos\theta = \frac{pc}{2E_1}$.

$$\therefore \quad \cos\theta = \frac{\sqrt{(1135)^2 - (135)^2}}{1135} = \frac{1127}{1135} = 0.9986$$

$$\therefore \quad \theta = 3.2° \text{ and } 2\theta = 6.4°.$$

Example 10 Determine the maximum kinetic energy of the electron produced by muon decay viz. $\mu = e + \nu + \bar{\nu}$. Take rest mass of muon as 105.5 MeV, electron rest mass 0.5 MeV. Neutrino and antineutrino have zero rest mass.

Solution

Maximum kinetic energy for the electron is possible when electron travels in one direction and the other two particles travel in the opposite direction.

Let M and m denote the rest masses of the muon and electron respectively. Then from conservation of energy

$$Mc^2 = \Gamma mc^2 + (E_1 + E_2) = \Gamma mc^2 + E \quad \text{say} \qquad \text{... (1)}$$

where $E = E_1 + E_2$ is the total energy carried by the neutrino and the antineutrino.

Since the muon is originally at rest, from conservation of momentum

$$\Gamma mv = \frac{E_1}{c} + \frac{E_2}{c} = \frac{E}{c} \qquad \text{... (2)}$$

Subtracting eqn. (2) from eqn. (1) gives

$$Mc^2 - \Gamma mvc = \Gamma mc^2$$

$$\therefore \quad M = \Gamma m \left(1 + \frac{v}{c}\right) = \frac{m(1 + v/c)}{(1 - v^2/c^2)^{1/2}} = m\sqrt{\frac{1+\beta}{1-\beta}}$$

$$\therefore \quad \frac{M}{m} = \sqrt{\frac{1+\beta}{1-\beta}}$$

squaring both sides and simplifying,

$$\beta = \frac{M^2 - m^2}{M^2 + m^2}$$

$$\therefore \quad \Gamma = \frac{1}{(1-\beta^2)^{1/2}} = \frac{M^2 + m^2}{2Mm}$$

Kinetic energy of the electron is $T = (\Gamma - 1)\, mc^2$

$$= \frac{(Mc^2 - mc^2)^2}{2Mc^2} = \frac{(105)^2}{2(105.5)} = 53.25 \text{ MeV}.$$

Example 11 A stationary electron of rest mass m_0 is struck by a photon of energy $E \gg m_0c^2$. Show that the maximum possible kinetic energy given to the electron is approximately $\left(E - \frac{m_0c^2}{2}\right)$.

Solution

Maximum possible kinetic energy is given to the electron when the photon is scattered backwards, i.e. its direction is reversed.

From conservation of energy we get

$$E + m_0c^2 = E' + mc^2 \qquad \text{... (1)}$$

where E' is the energy of the scattered photon and m is the mass of the scattered electron.

For conservation of momentum we must have

$$\frac{E}{c} + 0 = p - \frac{E'}{c} \text{ or } E + E' = pc \qquad \text{... (2)}$$

where p is the momentum of the scattered electron.

From Eqn. (2) $\qquad (E + E')^2 = p^2c^2 = m^2c^4 - m_0^2 c^4 \qquad$ ... (3)

From Eqn. (1), $\qquad (E - E')^2 = (mc^2 - m_0c^2)^2 \qquad$ ... (4)

Subtracting Eqn. (4) from Eqn. (3), we get

$$4EE' = 2m_0c^2\ (m - m_0)c^2$$

From Eqn. (1), kinetic energy of the electron is $mc^2 - m_0c^2 = (E - E')$. Hence above equation can be written as $4EE' = 2m_0c^2\ (E - E')$.

$$\therefore \qquad E' = \frac{m_0 c^2 E}{2E + m_0 c^2}$$

$\therefore$ Kinetic Energy of the electron is $E - E' = E - \dfrac{m_0 c^2 E}{2E + m_0 c^2}$.

$$\approx E - \frac{m_0 c^2 E}{2E} \qquad \because \quad m_0c^2 \ll E$$

$$= E - \frac{m_0 c^2}{2}.$$

Note also that energy of the backward scattered photon $E' \approx \dfrac{m_0 c^2}{2}$ = constant.

Example 12 Show that in case of Compton scattering from a free particle of rest mass m_0, the energy of the scattered photon is always less than $2m_0c^2$ if the angle through which the photon is scattered exceeds 60°. This is irrespective of how large the energy of the incident photon is and consequently, only high energy Compton scattered photons emerging out in a 60° cone can produce particle-antiparticle pairs.

Solution

Scattered wavelength is

$$\lambda' = \lambda + \frac{h}{m_0 c}\ (1 - \cos\theta)$$

Let $\theta > 60°$, then $\cos\ \theta < 0.5$

$$\therefore \qquad \lambda' > \lambda + \frac{h}{m_0 c}\ (0.5)$$

$$\therefore \qquad \frac{\lambda'}{hc} > \frac{\lambda}{hc} + \frac{1}{2 m_0 c^2}$$

$$\therefore \qquad \frac{1}{E'} > \frac{1}{E} + \frac{1}{2 m_0 c^2}$$

$$\therefore \qquad \frac{1}{E'} > \frac{E + 2m_0c^2}{E(2m_0c^2)}$$

$$\therefore \qquad E' < \frac{E(2m_0c^2)}{E + 2m_0c^2}$$

$$\therefore \qquad E' < \frac{2m_0c^2}{1 + 2m_0c^2/E}$$

$$\therefore \qquad E' < 2m_0c^2.$$

Hence only high energy Compton scattered photons emerging out in a 60° cone can produce particle-antiparticle pairs.

Example 13 Show that in Compton scattering the final energy of a photon of energy E scattered through an angle θ is given by $E' = \dfrac{E}{1 + \dfrac{E(1-\cos\theta)}{m_0c^2}}$. Also prove that the kinetic energy imparted to the electron is $T = \dfrac{\alpha}{1+\alpha}$ where $\alpha = \dfrac{\lambda' - \lambda}{\lambda}$.

Solution

$$\lambda' - \lambda = \frac{h(1-\cos\theta)}{m_0c}$$

$$\therefore \qquad \frac{c}{\nu'} - \frac{c}{\nu} = \frac{c(\nu - \nu')}{\nu\nu'} = \frac{h}{m_0c}(1 - \cos\theta)$$

$$\therefore \qquad \frac{\nu - \nu'}{\nu\nu'} = \frac{h}{m_0c^2}(1 - \cos\theta)$$

$$\therefore \qquad \frac{\nu - \nu'}{\nu'} = \frac{h\nu}{m_0c^2}(1 - \cos\theta) = \frac{E}{m_oc^2}(1 - \cos\theta)$$

$$\therefore \qquad \frac{\nu}{\nu'} = 1 + \frac{E}{m_0c^2}(1 - \cos\theta)$$

$$\therefore \qquad \frac{h\nu}{h\nu'} = 1 + \frac{E}{m_0c^2}(1 - \cos\theta) = \frac{E}{E'}$$

$$\therefore \qquad E = E'\left(1 + \frac{E}{m_0c^2}(1-\cos\theta)\right)$$

$$\therefore \qquad E' = \frac{E}{1 + \dfrac{E(1-\cos\theta)}{m_0c^2}}$$

Kinetic energy given to the electron is $T = E - E'$

$$\therefore \qquad T = E\left(1 - \frac{E'}{E}\right) = E\left(1 - \frac{\lambda}{\lambda'}\right)$$

Let $$\alpha = \frac{\lambda' - \lambda}{\lambda} = \frac{\lambda'}{\lambda} - 1. \text{ Then } \frac{\lambda'}{\lambda} = 1 + \alpha$$

$$\therefore \qquad T = E\left(1 - \frac{1}{1+\alpha}\right) = E\left(\frac{\alpha}{1+\alpha}\right)$$

Example 14 When a photon of energy E strikes an electron (rest mass m_0c^2) at rest, the electron is observed to be scattered through an angle ϕ. Show that its kinetic energy is

$$T = E\,\frac{2\alpha\cos^2\phi}{(1+\alpha)^2 - \alpha^2\cos^2\phi} \quad \text{where } \alpha = \frac{E}{m_0c^2}.$$

Solution

Refer to Fig. 7.7 (Compton Effect)

Let $E = h\nu$ and $E' = h\nu'$ = energy of the scattered photon. Kinetic energy given to the electron is $T = E - E'$.

From conservation of linear momentum, $\frac{E'}{c}\sin\theta = p\sin\phi$ or $E'\sin\theta = pc\sin\phi$ ... (1)

Also $\frac{E'}{c}\cos\theta + p\cos\phi = \frac{E}{c}$ or $E'\cos\theta = E - pc\cos\phi$ ... (2)

Squaring and adding equations (1) and (2);

$$E'^2 = E^2 + p^2c^2 - 2Epc\cos\phi$$

$$\begin{aligned}
\therefore \qquad 2Epc\cos\phi &= E^2 - E'^2 + p^2c^2 \\
&= (E + E')(E - E') + (m^2c^4 - m_0^2c^4) \\
&= (E + E')(E - E') + (mc^2 + m_0c^2)(mc^2 - m_0c^2) \\
&= (2E - T)T + (T + 2m_0c^2)(T)
\end{aligned}$$

because $$E - E' = T = mc^2 - m_0c^2$$

Above equation may be written as

$$T(E + m_0c^2) = Epc\cos\phi$$

$$\therefore \qquad T\left(E + \frac{E}{\alpha}\right) = Epc\cos\phi$$

$$\therefore \qquad T(1 + \alpha) = \alpha pc\cos\phi$$

$$\begin{aligned}
\therefore \qquad T^2(1 + \alpha)^2 &= \alpha^2\cos^2\phi\,(m^2c^4 - m_0^2c^4) \\
&= \alpha^2\cos^2\phi\,T(T + 2m_0c^2).
\end{aligned}$$

$$\therefore \qquad T(1 + \alpha)^2 = T\alpha^2\cos^2\phi + 2m_0c^2\alpha^2\cos^2\phi$$

$$\therefore \qquad T = \frac{2m_0c^2\,\alpha^2\cos^2\phi}{(1+\alpha)^2 - \alpha^2\cos^2\phi} = \frac{2\alpha\cos^2\phi}{(1+\alpha)^2 - \alpha^2\cos^2\phi}E$$

because $$m_0c^2 = \frac{E}{\alpha}$$

Example 15 A photon of energy E has a Compton collision with a stationary electron of rest mass energy m_0c^2. Due to collision the energy of the photon is reduced to half of its original energy and it is scattered in a direction which makes an angle of sixty degrees with its original direction. Find the value of E.

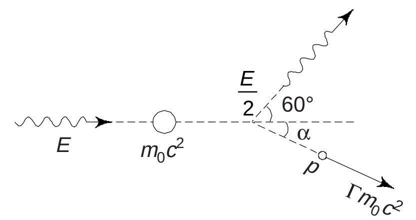

Fig. 7.19 For Illustrative Example 15

Solution

For conservation of energy we must have

$$E + m_0c^2 = \frac{E}{2} + \Gamma m_0c^2 \quad \text{or} \quad \Gamma = 1 + \frac{E}{2m_0c^2} \qquad (1)$$

From conservation of linear momentum we get

$$\frac{E}{c} = \frac{E}{2c}\cos 60° + p\cos\alpha \qquad \text{(See Fig. 7.19)} \quad ...\ (2)$$

$$\frac{E}{2c}\sin 60° = p\sin\alpha \qquad ...\ (3)$$

From eqn. (2) we get $pc\cos\alpha = \dfrac{3E}{4}$

From eqn. (3) we get $pc\sin\alpha = \dfrac{\sqrt{3}E}{4}$

Squaring and adding above two equations, we get

$$p^2c^2 = \frac{9E^2}{16} + \frac{3E^2}{16} = \left(\frac{3}{4}\right)E^2$$

Also $\qquad p^2c^2 = m^2c^4 - m_0^2c^4 = (\Gamma^2 - 1)\, m_0^2c^4$

$$\therefore \qquad \left(\frac{3}{4}\right)E^2 = (\Gamma^2 - 1)\, m_0^2c^4 = \left[\left(1 + \frac{E}{2m_0c^2}\right)^2 - 1\right] m_0^2c^4$$

$$\therefore \qquad \frac{3}{4}E^2 = m_0^2c^4 + \frac{E^2}{4} + Em_0c^2 - m_0^2c^4$$

On simplifying we get $E = 2m_0c^2$.

Example 16 A high energy particle A of rest mass m moving at speed $v_A = 0.6$ c relative to the laboratory frame S, collides with a target particle B, initially at rest and having rest mass 2 m. (a) What is the total energy E of the particles in frame S? (b) What is the velocity of the frame S' in which the total momentum of the two particles is zero? (c) What is the total energy of the particles in frame S'? (d) If the collision is inelastic what is the speed and the rest mass of the composite particle formed? Do the calculations in frame S as well as in frame S'.

Solution

We choose the positive X-axis in the direction of motion of the incident particle.

We then have for particle A;

rest mass = m; $\qquad \dot{x}_A = 0.6$ c; $\dot{y}_A = \dot{z}_A = 0$; $v_A = 0.6$ c

$$\Gamma_A = \left[1-\left(\frac{v_A}{c}\right)^2\right]^{-1/2} = \frac{5}{4};\ p_{xA} = \Gamma_A\, m\,\dot{x}_A = \frac{5}{4}\ \text{m (0.6 c)} = \frac{3}{4}\ \text{mc}$$

$$p_{yA} = p_{zA} = 0 \text{ and } E_A = \Gamma_A mc^2 = \frac{5}{4}\ mc^2$$

Similarly we have for particle B,

Rest mass $\quad m_B = 2\ \text{m};\ \dot{x}_B = \dot{y}_B = \dot{z}_B = 0,\ v_B = 0$

$$\Gamma_B = 1;\ P_{xB} = P_{yB} = P_{zB} = 0;\ E_B = \Gamma_B m_B c^2 = 2mc^2$$

(a) The total energy of the two particles in frame S is $E = E_A + E_B = \frac{5}{4}\ mc^2 + 2\ mc^2 = \frac{13}{4}\ mc^2$

(b) The momenta and energy in S as viewed in S' are given by $p_x' = \dfrac{p_x - vE/c^2}{\sqrt{1-v^2/c^2}}$, $p_y' = py$; $p_z' = pz$

and $E' = \dfrac{E - vp_x}{\sqrt{1-v^2/c^2}}$ where v is the velocity of the c. of m. of the system along the X-axis.

Momentum components of particles A and B in frame S' are

$$p_{xA}' = \frac{p_{xA} - vE_A/c^2}{\sqrt{1-v^2/\,^2}} = \frac{\frac{3}{4}mc - v\frac{5}{4}m}{\sqrt{1-v^2/c^2}};\ p_{yA}' = p_{zA}' = 0$$

$$p_{xB}' = \frac{p_{xB} - vE_B/c^2}{\sqrt{1-v^2/c^2}} = \frac{0 - v(2m)}{\sqrt{1-v^2/c^2}};\ p_{yB}' = p_{zB}' = 0$$

Total momentum of the system in the c. of m. frame S' is

$$P_{xA}' + p_{xB}' = \frac{\frac{3}{4}mc - v\frac{5}{4}m - 2mv}{\sqrt{1-v^2/c^2}} = 0$$

$$\therefore \qquad \frac{3c}{4} = \frac{13}{4}v \quad \text{or} \quad v = \frac{3}{13}c$$

Alternately $\quad \left|\overrightarrow{V_{\text{c.m.}}}\right| = \dfrac{c^2\left|\text{Total momentum } \vec{p} \text{ (in frame } S)\right|}{\text{Total energy in frame } S}$

$$= \frac{c^2\left(\frac{3}{4}mc\right)}{\frac{13}{4}mc^2} = \frac{3}{13}\text{c.}$$

(c) Total energy E' of the system in S' is $\left(E'_A + E'_B\right)$

Where $$E'_A = \frac{E_A - vp_{xA}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{5}{4}mc^2 - \left(\frac{3}{13}c\right)\left(\frac{3}{4}mc\right)}{\sqrt{1-\left(\frac{3}{13}\right)^2}} = \frac{7\,mc^2}{2\sqrt{10}}$$

$$= 1.107 \text{ mc}^2$$

and $$E'_B = \frac{E_B - vp_{xB}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\,mc^2 - 0}{\sqrt{1-\left(\frac{3}{13}\right)^2}} = \frac{13\,mc^2}{2\sqrt{10}} = 2.055 \text{ mc}^2$$

Total energy in frame S' is $E' = E'_A + E'_B$

$$= \frac{20\,\text{mc}^2}{2\sqrt{10}} = \sqrt{10}\,\text{mc}^2 = 3.162 \text{ mc}^2.$$

This is less than the total energy E in frame S. The difference between the two represents the energy carried by the centre of mass.

(d) Let M be the rest mass of the composite particle and v' its velocity in the frame S. Then conservation of total energy and momentum give respectively.

$$\Gamma mc^2 + 2mc^2 = \Gamma' Mc^2 \quad \text{i.e.} \quad (\Gamma + 2)mc^2 = \Gamma' Mc^2 \qquad \text{... (1)}$$

and $$\Gamma mv = \Gamma' Mv' \qquad \text{... (2)}$$

where $$\Gamma = \frac{1}{\sqrt{1-(0.6)^2}} = \frac{5}{4}$$

and $$\Gamma' = \frac{1}{\sqrt{1-v'^2/c^2}}$$

Dividing eqn. (2) by eqn. (1) we get

$$v' = \frac{\Gamma v}{\Gamma + 2} = \frac{(5/4)\,(0.6c)}{5/4+2} = \frac{3c}{13}$$

$\therefore$ $$\Gamma' = \frac{1}{\sqrt{1-9/169}} = \frac{13}{4\sqrt{10}}$$

From eqn. (2), $$M = \frac{\Gamma vm}{\Gamma' v'} = \frac{(5/4)\,(0.6c)\,m}{\left(13/4\sqrt{10}\right)\left(\frac{3c}{13}\right)} = \sqrt{10}\text{ m}$$

$$= 3.162 \text{ m.}$$

Let v_1 and v_2 denote the velocities of the particles A and B respectively as seen in frame S'. In this frame the two particles having equal and opposite momenta, collide to form a composite particle of rest mass M. The composite particle is at rest in S' by conservation of linear momentum.

For conservation of energy we must have

$$\Gamma_1 mc^2 + \Gamma_2(2m)c^2 = Mc^2 \quad \text{or} \quad M = (\Gamma_1 + 2\Gamma_2)\ m$$

where $$\Gamma_1 = \frac{1}{\sqrt{1-(v_1/c)^2}} \quad \text{and} \quad \Gamma_2 = \frac{1}{\sqrt{1-(v_2/c)^2}}.$$

In frame S', velocity of the particle B is equal to the velocity of the c. of m. as observed in frame S because particle B is at rest in frame S. Thus

$$v_2 = \frac{3c}{13} \quad \text{Hence } \Gamma_2 = \frac{13}{4\sqrt{10}}$$

From composition of velocities, velocity of particle A is given by $v = \dfrac{u-v}{1-uv/c^2}$,

that is $$v_1 = \frac{0.6c - 3/13c}{1-(0.6)\left(\dfrac{3}{13}\right)} = \frac{4.8c}{11.2} = \frac{3}{7}c$$

$\therefore$ $$\Gamma_1 = \frac{1}{\sqrt{1-(3/7)^2}} = \frac{7}{2\sqrt{10}}$$

$\therefore$ $$M = m(\Gamma_1 + 2\Gamma_2) = \frac{7m}{2\sqrt{10}} + \frac{13m}{2\sqrt{10}} = \sqrt{10}\ m \quad \text{as before.}$$

Example 17 A body (rest mass m) travelling at a speed v is incident on an identical stationary body. (a) Show that the c. of m. of the body system has a velocity $\Gamma v/\Gamma + 1$ where $\Gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$. (b) If the collision is perfectly inelastic show that the rest mass of the composite body is $M = m\sqrt{2(\Gamma+1)}$.

Solution

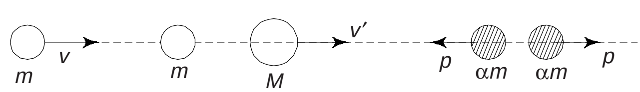

Fig. 7.20 For Illustrative Examples 17 and 18

By conservation of energy, $\Gamma\ mc^2 + mc^2 = \dfrac{Mc^2}{\sqrt{1-v'^2/c^2}}$ where v' is the velocity of the composite body in the laboratory frame.

$\therefore$ $$(\Gamma + 1)\ mc^2 = \frac{Mc^2}{\sqrt{1-v'^2/c^2}} \qquad \text{... (1)}$$

By conservation of momentum, $$\Gamma\ mv = \frac{Mv'}{\sqrt{1-v'^2/c^2}} \qquad \text{... (2)}$$

Dividing eqn. (2) by eqn (1); $\dfrac{\Gamma v}{\Gamma + 1} = v'$.

In the c. of m. frame, the composite body is stationary, therefore $v' = \Gamma v/\Gamma + 1$ is the velocity of the c. of m. of the system.

From eqn. (1); $M = (\Gamma + 1)\ m\ \sqrt{1 - v'^2/c^2}$

$$= (\Gamma + 1)\ m\ \sqrt{1 - \frac{\Gamma^2}{(\Gamma+1)^2} \cdot \frac{v^2}{c^2}} = (\Gamma + 1)\ m\ \sqrt{\frac{(\Gamma+1)^2 c^2 - \Gamma^2 v^2}{(\Gamma+1)^2 c^2}}$$

$$= m\sqrt{(\Gamma+1)^2 - \Gamma^2 v^2/c^2} = m\sqrt{(\Gamma+1)^2 - \Gamma^2 \frac{(\Gamma^2 - 1)}{\Gamma^2}}$$

because $\qquad v^2/c^2 = \left(\dfrac{\Gamma^2 - 1}{\Gamma^2}\right)$

$$\therefore \qquad M = m\sqrt{(\Gamma+1)^2 - (\Gamma^2 - 1)} = m\sqrt{2(\Gamma+1)}$$

Example 18 A body (rest mass m) travelling with a velocity v, collides with an identical body at rest to form a composite body. The composite body subsequently breaks into two particles each of mass αm. Show that the momentum of each particle in the zero momentum frame is given by

$$p = mc\left[\frac{(1 + \Gamma - 2\alpha^2)}{2}\right]^{1/2} \quad \text{where } \Gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Solution From above example, the composite particle has rest mass $M = m\sqrt{2(\Gamma+1)}$ and travels at velocity $v' = \left(\dfrac{\Gamma v}{\Gamma + 1}\right)$ as observed in the laboratory frame.

In the zero momentum frame, the composite particle is at rest and its total energy is Mc^2.

$\therefore$ Total energy of each particle is $\dfrac{Mc^2}{2} = \dfrac{mc^2\sqrt{2(\Gamma+1)}}{2}$

$$\therefore \qquad \left[\frac{mc^2}{2}\sqrt{2(\Gamma+1)}\right]^2 = p^2c^2 + \alpha^2 m^2 c^4$$

$$\therefore \qquad p^2c^2 = m^2c^4\left[\frac{\Gamma+1}{2} - \alpha^2\right]$$

$$\text{or} \qquad p = mc\left(\frac{1 + \Gamma - 2\alpha^2}{2}\right)^{1/2}.$$

Maximum value of α obtains when $p = 0$, that is when $\alpha = \left(\frac{\Gamma+1}{2}\right)^{1/2}$. In this case, the composite particles breaks into two particles which are at rest in the zero momentum frame. There is no kinetic energy in the zero momentum frame. This is the case of maximum conversion of original kinetic energy into mass.

Example 19 A particle of rest mass m having kinetic energy T is incident on a stationary particle of rest mass M. As a result of the collision the incident particle is scattered through 90° and has a momentum P. Show that after the collision the target particle has momentum $(P^2 + 2Tm + T^2/c^2)^{1/2}$ and travels in a direction that makes angle $\theta = \tan^{-1}\left(\frac{P}{(2Tm + T^2/c^2)^{1/2}}\right)$ with the direction of the incident particle.

Solution Let p be the momentum of the incident particle and p' the momentum of the target particle after the collision. See Fig. 7.21. By conservation of momentum, $P = p'\sin\theta$

$$p = p' \cos\theta$$

$$\therefore \quad \tan\theta = P/p.$$

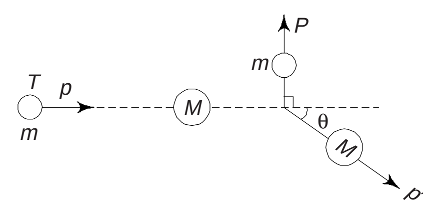

Fig. 7.21 For Illustrative Example 19

Total energy of the incident particle is

$$E = mc^2 + T = \sqrt{p^2c^2 + m^2c^4}$$

$$\therefore \quad m^2c^4 + 2mTc^2 + T^2 = p^2c^2 + m^2c^4$$

$$\therefore \quad p = \sqrt{2mT + T^2/c^2}$$

$$\therefore \quad \theta = \tan^{-1} \frac{P}{(2mT + T^2/c^2)^{1/2}} \quad \text{as required.}$$

Also

$$P^2 = p'^2 \sin^2\theta$$

$$p^2 = p'^2 \cos^2\theta$$

$$\therefore \quad p'^2 = P^2 + p^2 = P^2 + 2mT + T^2/c^2$$

$$\therefore \quad p' = (P^2 + 2mT + T^2/c^2)^{1/2}$$

Example 20 A particle of rest mass m and kinetic energy $2mc^2$ strikes and sticks to a stationary particle of rest mass 2 m. What is the rest mass of the composite particle so formed? Compare the kinetic energy of the system before and after the collision.

Solution Total energy of the incident particle = rest mass energy mc^2 + kinetic energy $2mc^2 = 3mc^2 = E_i$ say.

Let p be the momentum of the incident particle.

Then

$$E_i^2 = p^2c^2 + m^2c^4 \text{ or } p^2c^2 = E_i^2 - m^2c^4 = 9m^2c^4 - m^2c^4$$

$$= 8m^2c^4.$$

Total energy of the system before collision is

$$3mc^2 + 2mc^2 = 5mc^2 = E \text{ say.}$$

Let M be the rest mass of the composite particle. By conservation of linear momentum, its linear momentum is p. Let E' denote the total energy of the composite particle. By conservation of total energy we must have

$$E = E' \quad \text{or} \quad E^2 = E'^2$$

$$\therefore \quad E^2 = 25m^2c^4 = E'^2 = M^2c^4 + p^2c^2 = M^2c^4 + 8\,m^2c^4$$

$$\therefore \quad M^2c^4 = 17m^2c^4 \quad \text{or} \quad M = \sqrt{17}\,m.$$

Kinetic energy of the composite particle = $E' - Mc^2$

$$= 5mc^2 - \sqrt{17}\,mc^2 = (5 - 4 \cdot 123)mc^2 = 0.877mc^2.$$

$$\therefore \qquad \frac{\text{K.E. of system before collision}}{\text{K.E. of system after collision}} = \frac{2mc^2}{0.877} > 1.$$

Example 21 It is possible that an electron-positron pair is produced when a photon of sufficiently large energy strikes an electron which for all practical purposes can be considered stationary. What is the minimum photon energy required for this? Electron/positron rest mass = 0.51 MeV/c^2.

Solution Let E be the minimum photon energy required. Then in frame S' moving with the c. of m. of the system, the end products are stationary. Total energy is $E' = 3mc^2$ where mc^2 = rest mass energy of electron/positron. Total momentum is zero since all three particles are stationary.

In laboratory frame S, initial total energy is $(E + mc^2)$ and initial momentum is $\frac{E}{c}$.

$$\therefore \qquad (3mc^2) - (0)^2 = (E + mc^2)^2 - c^2\left(\frac{E}{c}\right)^2$$

$$\therefore \qquad E = 4mc^2 = 2.04 \text{ MeV}.$$

Example 22 A particle of rest mass m_1 and total energy E_1 travelling with a velocity $v \approx c$ collides with a stationary particle of rest mass m_2. Show that (a) the energy available in the c. m. system is $E' \approx (2E_1 m_2 c^2)^{1/2}$ and (b) velocity of the c. of m. system is $v_{\text{c.m.}} \approx c\left[1 - \frac{m_2 c^2}{E_1}\right]$.

Solution Let E and E' denote the total energies and p and p' the total momenta in frames S and S' respectively.

Then $\qquad E^2 - p^2c^2 = E'^2 - p'^2c^2$

In laboratory frame S, $\qquad E = E_1 + m_2c^2$ and

$$p^2c^2 = E_1^2 - m_1^2c^4.$$

In the c. of m. frame S', $p' = 0$

$$\therefore \quad (E_1 + m_2c^2)^2 - \left(E_1^2 - m_1^2c^4\right) = E'^2$$

$$= 2m_2c^2E_1 + m_1^2c^4 + m_2^2c^4$$

$$\therefore \qquad E' = \left(2m_2c^2E_1 + m_1^2c^4 + m_2^2c^4\right)^{1/2}$$

$$\approx (2m_2c^2E_1)^{1/2} \quad \because \quad E_1 \gg \left(m_1^2c^4 + m_2^2c^4\right).$$

Also $\qquad v_{\text{c.m.}} = \frac{pc^2}{E} = \frac{c\left(E_1^2 - m_1^2c^4\right)^{1/2}}{E_1 + m_2c^2} \approx \frac{cE_1}{E_1 + m_2c^2}$

$$= \frac{cE_1}{E_1\left(1+\frac{m_2c^2}{E_1}\right)} = c\left(1+\frac{m_2c^2}{E_1}\right)^{-1} = c\left(1-\frac{m_2c^2}{E_1}\right)$$

because $E_1 \gg m_2c^2$ that is $\frac{m_2c^2}{E_1} \ll 1$.

Example 23 A photon hits a stationary proton to produce a pion in the reaction $\gamma + p \to n + \pi^+$. What is the minimum photon energy required. Take neutron/proton mass = 1836 m_e; m_π = 273m_e and m_ec^2 = 0.511 MeV.

Solution Let, E, E' and p, p' denote respectively the total energy and linear momentum of the system in laboratory frame S and c. of m. frame S'. Then

$$E^2 - p^2c^2 = E'^2 - p'^2c^2.$$

In frame S, $$p = \frac{h\nu}{c} \text{ and } E = h\nu + mc^2$$

In frame S', $p' = 0$ and $E' = mc^2 + m_\pi c^2$ when the products of reaction are stationary.
m = rest mass of proton/neutron.

$$\therefore \quad (h\nu + mc^2)^2 - (h\nu)^2 = (mc^2 + m_\pi c^2)^2 - 0$$

$$\therefore \quad 2h\nu mc^2 + m^2c^4 = m^2c^4 + m_\pi^2c^4 + 2mm_\pi c^4$$

$$\therefore \quad h\nu = \frac{(m_\pi c^2)^2}{2mc^2} + m_\pi c^2 = \left[\frac{(273)^2}{2\times 1836} + 273\right] m_ec^2$$

$$= \left(\frac{(273)^2}{3672} + 273\right) 0.511 \text{ MeV} = 150 \text{ MeV}.$$

Example 24 Neutral pions are produced by collision of high energy protons with target protons (stationary) in liquid hydrogen according to the reaction $p + p \to p + p + \pi^\circ$. Find the minimum kinetic energy required of the incident proton beam. Take m_p = 1836 m_e; m_{π° = 264 m_e and m_ec^2 = 0.511 MeV.

Solution When the kinetic energy of the incident proton beam is minimum possible for the reaction to proceed, all the product particles will be at rest in the c. of m. frame. Therefore they are observed to move with velocity $V_{\text{c.m.}}$ in the laboratory frame.

In the laboratory frame, the total energy of the system before collision is
$E = \Gamma m_pc^2 + m_pc^2 = (\Gamma + 1)\, m_pc^2$ where $\Gamma = (1 - v^2/c^2)^{-1/2}$, v being the incident proton velocity. Momentum of the system in the laboratory is $P = \Gamma m_pv$. Hence

$$V_{\text{c.m.}} = \frac{c^2P}{E} = \frac{\Gamma}{\Gamma+1}\, v$$

Hence $$\Gamma_{\text{c.m.}} = \left(1-\frac{v_{\text{c.m.}}^2}{c^2}\right)^{-1/2} = \left(\frac{2}{1+\Gamma}\right)^{-1/2}.$$

Energy of the system of particles in the laboratory frame after the reaction is $E' = (2m_p + m_{\pi^\circ})\, c^2\, \Gamma_{\text{c.m.}}$

This must be equal to E, the energy of the system in the laboratory frame before the reaction.

$$\therefore \quad (\Gamma + 1)m_pc^2 = (2m_p + m_{\pi^\circ})c^2\,\Gamma_{\text{c.m}} = (2m_p + m_{\pi^\circ})c^2\left(\frac{\Gamma+1}{2}\right)^{1/2}$$

$$\therefore \quad \sqrt{2(\Gamma+1)}\;m_p = (2m_p + m_{\pi^\circ})$$

squaring both sides and simplifying we get

$$\Gamma = \frac{1}{2}\left(2+\frac{m_{\pi^\circ}}{m_p}\right)^2 - 1$$

For this value of Γ, the kinetic energy of the incident proton is

$$T = (\Gamma - 1)m_pc^2 = \left[\frac{1}{2}\left(2+\frac{m_{\pi^\circ}}{m_p}\right)^2 - 2\right] m_pc^2$$

On simplifying this we get

$$T = m_{\pi^\circ}c^2\left(2+\frac{m_{\pi^\circ}}{2m_p}\right) = 264m_ec^2\left(2+\frac{264}{3672}\right)$$

$$= (264)\,(0.511)\left(2+\frac{264}{3672}\right)\text{ MeV} = 279.6\text{ MeV.}$$

Example 25 A photon strikes a stationary electron to produce an electron positron pair in the process $\gamma + e^- \rightarrow e^- + e^+ + e^-$ (recoiling electron). What is the minimum photon energy required?

Solution Let $h\nu$ be the minimum photon energy in the laboratory frame required for the process. Let E and p denote the total energy of the system and its momentum respectively in the laboratory frame.

Let E' and p' be the total energy and momentum of the system respectively after the process in the c. of m. frame. Then $p' = 0$ when the incident photon energy is the minimum because the products of the reaction are stationary.

Now $E = h\nu + mc^2$ and $p = \dfrac{h\nu}{c}$ where mc^2 is the rest mass energy of electron/positron.

Also $$E' = 3mc^2 \text{ and } p' = 0$$

Since $$E^2 - p^2c^2 = E'^2 - p'^2c^2,$$

$$(h\nu + mc^2)^2 - c^2\left(\frac{h\nu}{c}\right)^2 = (3mc^2)^2$$

On simplifying we get $h\nu = 4mc^2$. Hence minimum photon energy required is four times the rest mass energy of an electron. This is twice the minimum photon energy for pair production in the vicinity of a heavy nucleus.

Example 26 In nuclear physics, the Q value of a reaction is determined from measurements of the kinetic energies T_x and T_y of the incident and emerging particles x and y at a known angle θ between the directions of x and y. Show that the Q value for low energy reactions is

$$Q = T_y\left(1+\frac{m_y}{M_Y}\right)-T_x\left(1-\frac{m_x}{M_Y}\right)-2\frac{\sqrt{m_x\, m_y\, T_x\, T_y}}{M_Y}\cos\theta$$

in the standard notation used for the nuclear reaction $X\ (x, y)Y$.

Solution As shown in Fig. 7.22(a), the bombarding particle x (rest mass m_x, momentum $\vec{p}_x$) strikes the stationary target nucleus X(rest mass M_X). As a result of the nuclear reaction, the emergent light particle y(rest mass m_y, momentum $\vec{p}_y$) travels as shown while the product nucleus Y(rest mass M_Y) is left with momentum $\vec{P}_Y$. For momentum conservation $\vec{p}_x = \vec{p}_y + \vec{P}_Y$ as shown in the momentum triangle in Fig. 7.22 (b).

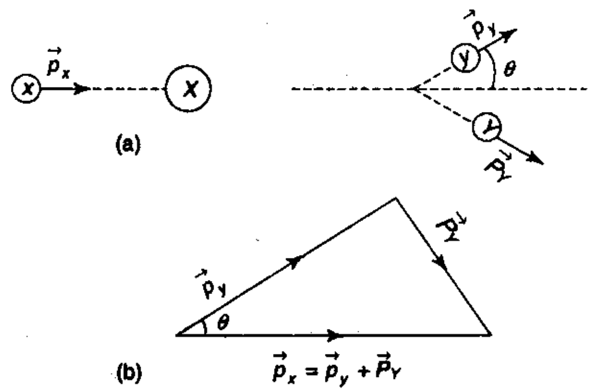

Fig. 7.22 For Illustrative Example 26

From the momentum triangle,

$P_Y^2 = p_y^2 + p_x^2 - 2p_xp_y\cos\theta$ where θ is the angle between directions of travel of x and y.

For low energies, we can use the simple relation (momentum $mv)^2 =$ $\left(\text{Kinetic energy } \frac{1}{2}\, mv^2\right)$ (2 mass m).

Hence above equation can be written as

$$2M_YT_Y = 2m_yT_y + 2m_xT_x - 2\sqrt{2m_x\, T_x\, 2m_y\, T_y}\ \cos\theta$$

$$\therefore \qquad T_Y = \frac{m_yT_y + m_xT_x}{M_Y} - 2\frac{\sqrt{m_xT_xm_yT_y}}{M_Y}\cos\theta$$

$$\therefore \qquad Q = T_Y + T_y - T_x$$

$$= \frac{m_yT_y + m_xT_x}{M_Y} + T_y - T_x - 2\frac{\sqrt{m_xT_xm_yT_y}}{M_Y}\cos\theta$$

$$= T_y\left[1+\frac{m_y}{M_Y}\right]-T_x\left[1-\frac{m_x}{M_Y}\right]-2\frac{\sqrt{m_xT_xm_yT_y}}{M_Y}\cos\theta$$

Example 27 Two protons each of kinetic energy T, collide head on. All the initial kinetic energy viz. $2T$ can be available to create rest mass. Suppose the same rest mass ($2T/c^2$) is to be created by a proton striking a stationary target proton. What is the minimum kinetic energy required of the incident proton? Find its value if T = 25 GeV. Take rest mass of proton m_0 = 1 GeV/c^2.

Solution Rest mass of all particles entering and leaving the reaction is $\left(2m_0 + 2m_0 + \dfrac{2T}{c^2}\right)$ and $Q = -2T$.

Hence threshold energy is

$$\frac{2T\,(4m_0 + 2T/c^2)}{2m_0} = 2T\left(2 + \frac{T}{m_0 c^2}\right)$$

$$= 2(25)\left(2 + \frac{25}{1}\right) = 50 \times 27 = 1350 \text{ GeV}.$$

EXERCISES

1. An unstable stationary particle of rest mass M decays spontaneously into two particles of rest mass m_1 and m_2. Determine the energies of the decay products.
2. A photon carrying energy Q is incident on an atom of rest mass M_0 and is completely absorbed by it. Show that the atom recoils at a speed of $\dfrac{c}{\left(1 + \dfrac{M_0 c^2}{Q}\right)}$.
3. A stationary nucleus of rest mass M_0 is in an excited state and has excitation energy equal to Q_0. The nucleus returns to the ground state by emitting a photon of energy Q. Show that
$$Q = Q_0\left(1 - \frac{Q_0}{2M_0 c^2}\right).$$
4. What is pair production? Can a photon disappear in free space and create a particle-antiparticle pair? Explain. Show that pair production is possible when a suitable particle such as an atomic nucleus participates in the process. What is the role of such a particle?
5. What is pair annihilation? What is the energy of the photons produced in the simplest process of pair annihilation?
6. What is Compton scattering? How is it explained? Derive an expression for the change in wavelength of a photon when it is deflected through an angle θ due to Compton scattering on encountering a stationary electron.
7. A photon of energy E collides with a stationary electron of rest mass m_0. After the collision the direction of photon's travel is deflected through an angle θ and its energy reduces to E'. Show that $m_0c^2\left(\dfrac{1}{E'} - \dfrac{1}{E}\right) = 1 - \cos\theta$. Hence show that the wavelength of the photon increases by $\Delta\lambda = \dfrac{2h}{m_0 c}$ $\sin^2 \theta/2$ where h is Planck's constant.
8. Treat the symmetrical elastic collision of identical particles relativistically. Compare its outcome with Newtonian mechanics according to which the two particles depart at right angles to each other.
9. A particle of rest mass m_0, energy e_0 and momentum p_0 has an elastic head-on collision with a stationary mass M_0 which is knocked straight forward with energy E and momentum P. The energy and momentum of the projectile after the collision are denoted respectively by e and p. Obtain expressions for p and P in terms of initial momentum and rest masses of particles participating in the collision. Compare the results with those of Newtonian mechanics.

10. Inelastic collision of a sufficiently energetic proton with a stationary proton can result in proton-antiproton pair production viz. $p + p \rightarrow p + p + p + \bar{p}$. Study such a collision in a frame of reference in which the total linear momentum of the system of particles is zero before the collision. Hence determine the minimum kinetic energy to which the incident proton (rest mass energy 938 MeV) beam must be accelerated for production of proton-antiproton pair in the laboratory.
11. A projectile particle (rest mass m_1) collides inelastically with a stationary particle (rest mass m_2). After collision, rest mass of the system increases by Δm. Show that the kinetic energy required of the projectile particle is $T = \Delta mc^2 \left[1 + \frac{m_1}{m_2} + \frac{\Delta m}{2m_2}\right]$.
12. What do you mean by Q of a nuclear reaction? What are endoergic and exoergic nuclear reactions? What is meant by threshold energy for an endoergic nuclear reaction? Show that

$$E_{th} = |Q| \frac{(m_x + M_X + m_y + M_Y)}{2M_x}$$ where the symbols have their usual meanings.

13. A stationary shell of rest mass M explodes into two fragments of rest masses M_1 and M_2. Show that their respective energies are given by

$$E_1 = \frac{M^2 + M_1^2 - M_2^2}{2M} c^2; \; E_2 = \frac{M^2 - M_1^2 + M_2^2}{2M} c^2.$$

14. A stationary nucleus of rest mass M is in an excited state with excitation energy E. If it returns to ground state by emitting a photon of frequency f, show that $f = \frac{E}{h}\left(1 - \frac{E}{2Mc^2}\right)$.
15. A photon of energy E has a Compton type head-on collision with an approaching electron of rest mass m. As a result of the collision the photon recoils straight backward with the same energy. Show that the electron velocity is $\dfrac{c}{\left[1 + \left(\dfrac{mc^2}{E}\right)^2\right]^{1/2}}$.
16. Show that a free electron at rest cannot absorb an incident photon.
17. A photon of energy E is incident on a stationary mass M. What is the velocity of the c. of m. of the system? $\left(Ans.\ v = \dfrac{Ec}{E + Mc^2}\right)$
18. A photon whose energy is equal to the rest mass energy of an electron collides with a stationary electron in the neighbourhood of a massive nucleus. As a result of the collision the photon is completely absorbed giving almost all of its energy to the electron so that the energy given to the nucleus is negligible. Find the velocity of the electron after the collision. $\left(Ans.\ v = \dfrac{c\sqrt{3}}{2}\right)$
19. A body of rest mass m and velocity v = 0.6 c collides with another similar body at rest. After the collision, the two bodies coalesce to form a composite body. Calculate the rest mass and velocity of the composite body so formed. $\left(Ans.\ \sqrt{\dfrac{9}{2}}m, \dfrac{c}{3}\right)$

20. A particle of rest mass m and kinetic energy nmc^2 collides inelastically with a stationary particle of rest mass Nm. Calculate the speed and the rest mass of the composite particle formed.

$$\left(Ans.\ v = \frac{\sqrt{n(n+2)}}{1+n+N}\, c,\ M = m\sqrt{(1+N)^2 + 2nN} \right)$$

21. An antiproton of kinetic energy 2/3 GeV travelling along positive X direction strikes a stationary proton. Their annihilation results into two photons; one travelling forward and the other backward with respect to the incident antiproton. Find their energies (a) as measured in the laboratory and (b) as measured in the rest frame of the incident antiproton. Rest mass energy of each particle is 1 GeV.

(*Ans*. (a) 2 GeV +ve X-direction, $\frac{2}{3}$ GeV –ve X-direction

(b) 2 GeV –ve X-direction, $\frac{2}{3}$ GeV +ve X-direction)

22. Show that the following processes are dynamically impossible:
    (a) A single photon in empty space is transformed into an electron and positron.
    (b) A fast positron and a stationary electron annihilate producing only one photon.

    Hint: Write equations for conservation of energy and momentum and show that they are mutually inconsistent.

23. As a result of Compton collision with an electron, the incident photon is scattered through an angle of 60° and carries only half of its initial energy. What is the frequency of the incident photon? Take

$$\frac{\text{Planck's constant}}{\text{Rest mass of electron}} = 7.2 \times 10^{-2}\ \text{m}^2/\text{s}.$$ (*Ans*. 2.5×10^{18} Hz.)

24. Consider the collision of a photon with a stationary atom in an excited state. After the collision, the incident photon retraces its path with unchanged energy leaving the atom in its ground state. Show that the energy of excitation of the atom is given by

$$m_0c^2 \left\{ \left[1 + \left(\frac{2E}{m_0 c^2} \right)^2 \right]^{1/2} - 1 \right\}$$ where E is the photon energy and m_0 = rest mass of the atom.

25. A particle (of rest mass 500 MeV) in flight decays into two particles (each of rest mass 125 MeV). If one of them is found to be stationary what is the kinetic energy of the other? What is the kinetic energy of the parent particle? (*Ans*. Daughter 750 MeV; Parent 500 MeV)

26. An accelerator is designed to accelerate protons to 300 GeV. What is the maximum possible rest mass of particle X that can be produced when a proton beam of above energy is incident on a stationary proton in the reaction $p + p \to p + p + X$. Take proton mass as 1 GeV. (*Ans*. 22.5 GeV)

27. Calculate the maximum kinetic energy of the electron produced by muon decay viz. $\mu \to e$ + neutrino + antineutrino. Take rest mass of muon as 105.6 MeV, that of electron as 0.51 MeV. Neutrinos and antineutrinos have zero rest mass.

28. Show that in e^-, e^+ annihilation, atleast two photons must be produced. What are the energies of the two photons, if the electron and the positron are at rest when they annihilate each other?

(*Ans*. 0.51 MeV)

29. Calculate the threshold energies in MeV for the following processes?
    (a) $\gamma + p \to p + \pi^\circ$
    (b) $\pi^- + p \to \pi^+ + \pi^- + n$
    (c) $p + p \to p + p + p + \bar{p}$

    Assume that rest mass of nucleon = 940 MeV/c^2.

Rest mass of charged pions = 140 MeV.
Rest mass of neutral pion = 135 MeV.

(*Ans.* (a) 144.5 MeV; (b) 171.1 MeV; (c) 5640 MeV)

30. A stationary pion (rest mass M) decays into a muon (rest mass m) and a massless neutrino. Show that the speed of the muon is given by $v = c\,\dfrac{(M/m)^2 - 1}{(M/m)^2 + 1}$.

31. A pion (rest mass M) travelling with velocity v decays in flight into a muon (rest mass m) and a massless neutrino as shown in Fig. 7.23. Show that

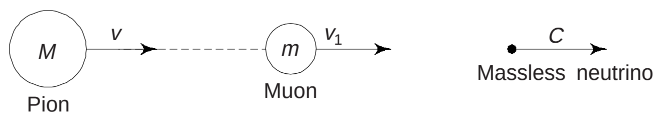

Fig. 7.23 For Exercise 31

$$v_1 = c(1 - \eta)/1 + \eta \quad \text{where}$$

$$\eta = \left(\frac{M}{m}\right)^2 \frac{1 - v/c}{1 + v/c}$$

Under what circumstances will v_1 be (a) nearly equal to c? (b) equal to zero? (c) negative?

32. Protons are accelerated to high energies in order to study inelastic interactions in proton-proton collisions using target protons essentially at rest in the laboratory. Estimate the fraction of proton energy wasted as they are progressively given energies of (a) 3 GeV; (b) 10 GeV, (c) 25 GeV; (d) 200 GeV and (e) 1000 GeV. Take proton rest mass energy to be 1 GeV.

(*Ans.* (a) 0.6; (b) 0.71; (c) 0.8; (d) 0.91; (e) 0.96.)

Suggested Further Reading

Rosser W.G.V.: *An Introduction To The Theory Of Relativity* (Butterworth London).
A.P. French: *Special Relativity* (ELBS).
H. Semat and R. Katz: *Physics* (Holt Rinehart and Winston—New York).
A.A. Barlett, "*Compton Effect*" Am. J. Phys. **32**, 120(1964).
A. H.Compton, "*The Scattering Of X-rays As Particles*", Am. J. Phys. **29**, 817(1961).
H. Lustig, "*The Mossbauer Effect*" Am. J. Phys. **29**, 1(1961).
F. Reines and C.L. Cowan Jr., "*The Neutrino*", Nature **178**, 446(1956).

Chapter

8 Four-Vectors

The theory of special relativity can be developed in a very concise and yet elegant way by using the so called concept of four-vectors. It is a conceptual extension of ordinary three-dimensional vectors to four-dimensional vectors.

8.1 SPACE-TIME CONTINUUM (MINKOWSKI SPACE)

Lorentz Transformation (L.T.) and its consequences suggest that the space coordinates (x, y, z) and time (t) should be treated uniformly in the theory of relativity. Minkowski suggested that the ordinary three-dimensional space plus the time be considered as four-dimensional continuum or Minkowski space.

Just as the ordinary three dimensional space can be labelled by three coordinates x, y and z; it is convenient to think of a four dimensional space-time labelled by the four coordinates x, y, z and t. For dimensional uniformity, the fourth coordinate may be taken as ct or preferably as ict ($i = \sqrt{-1}$) to remind us that space and time are not quite the same or equivalent.

Instead of variables x, y, z and t in frame S, we now introduce the variables X_μ (μ = 1, 2, 3, 4) given by

$$X_1 = x;\ X_2 = y;\ X_3 = z \text{ and } X_4 = ict \qquad \text{... (1a)}$$

The corresponding quantities in frame S' are X'_μ (μ = 1, 2, 3, 4); that is

$$X'_1 = x';\ X'_2 = y';\ X'_3 = z' \text{ and } X'_4 = ict' \qquad \text{... (1b)}$$

From L.T.; since $x' = \Gamma(x - vt)$ . . . etc.,

$$X'_1 = \Gamma\left(X_1 - \frac{vX_4}{ic}\right) = \Gamma\left(X_1 + \frac{iv}{c}X_4\right) \qquad \text{... (2)}$$

$$X'_2 = X_2;\ X'_3 = X_3 \qquad \text{... (3)}$$

Also since $$t' = \Gamma\left(t - \frac{vx}{c^2}\right)$$

$$ict' = \Gamma\left(ict - \frac{ivx}{c}\right)$$

$\therefore$ $$X'_4 = \Gamma\left(X_4 - \frac{ivX_1}{c}\right) \qquad \text{... (4)}$$

The inverse relations are easily written down by changing the sign of v. Thus

$$X_1 = \Gamma\left(X_1' - \frac{ivX_4'}{c}\right) \quad \text{... (5a)}$$

$$X_2 = X_2' \,;\; X_3 = X_3' \quad \text{... (5b)}$$

And $$X_4 = \Gamma\left(X_4' + \frac{ivX_1'}{c}\right) \quad \text{... (5c)}$$

Note that the above equations are just Lorentz transformations for X_1, X_2, X_3, X_4 just as equations (6) and (7) of chapter 2 are L.T. for coordinates x, y, z and t.

8.2 FOUR-VECTORS

A four-vector A_μ is a set of four quantities (A_1, A_2, A_3, A_4) which transform under a Lorentz Transformation in the same way as the X_1, X_2, X_3, X_4 coordinates of a point in the four dimensional space-time continuum. Thus

$$\left.\begin{aligned} A_1' &= \Gamma\left(A_1 + \frac{ivA_4}{c}\right);\; A_2' = A_2;\; A_3' = A_3 \\ \text{and} \quad A_4' &= \Gamma\left(A_4 - \frac{ivA_1}{c}\right) \end{aligned}\right\} \quad \text{... (6)}$$

The first three components (A_1, A_2 A_3) are the components of an ordinary three-dimensional vector $\vec{A}$ just as x, y, z are the components of $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$

The sum of two four-vectors is also a four-vector so that $C_\mu = (A + B)_\mu = A_\mu + B_\mu$.

Thus $\quad C_1 = (A_1 + B_1);\; C_2 = (A_2 + B_2);\; C_3 = (A_3 + B_3);\; C_4 = (A_4 + B_4)$

The scalar product of two four-vectors is defined by extending the scalar product ($\vec{A}\cdot\vec{B}$) of two ordinary vectors $\vec{A}$ and $\vec{B}$. Thus scalar product of two four-vectors A and B

$$= \Sigma_\mu A_\mu B_\mu = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4.$$

The square of the 'length' of a four-vector is defined as the scalar product of the four-vector with itself.

Thus $\quad (\text{length})^2 = \Sigma_\mu A_\mu^2 = A_1^2 + A_2^2 + A_3^2 + A_4^2$

$$= \Gamma^2\left(A_1' - \frac{ivA_4'}{c}\right)^2 + A_2'^2 + A_3'^2 + \Gamma^2\left(A_4' - \frac{ivA_1'}{c}\right)^2$$

$$= A_1'^2 + A_2'^2 + A_3'^2 + A_4'^2 = \Sigma_\mu A_\mu'^2 \text{ because } \Gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1.$$

We now see that the length of a four-vector is unchanged by a L.T. It is an invariant. An invariant is a quantity whose value does not change in a L.T. For example, the dot product of two four-vectors ($\Sigma_\mu A_\mu B_\mu$) is an invariant. 'Length' of a four-vector is a special case of dot product or scalar product of two vectors.

If all the components of a four-vector are multiplied by a scalar (α) an invariant, then we get a new four-vector of length α times the original length.

8.3 SOME IMPORTANT FOUR-VECTORS

(A) Position Four-Vector

The four-vector X with components $X_1 = x$; $X_2 = y$; $X_3 = z$, $X_4 = ict$ (introduced above) is called position four-vector.

(B) Velocity Four-Vector

Let X and $X + dX$ be the position four-vectors of a particle at times t and $t + dt$ respectively. Then $dX = (dX_1, dX_2, dX_3, dX_4)$ is a four vector. If this four-vector is multiplied by the invariant $\frac{1}{d\tau}$ we get another four-vector namely

$$U = \left(\frac{dX_1}{d\tau}, \frac{dX_2}{d\tau}, \frac{dX_3}{d\tau}, \frac{dX_4}{d\tau}\right)$$

$$= \left(\frac{dx}{dt\sqrt{1-u^2/c^2}}; \frac{dy}{dt\sqrt{1-u^2/c^2}}; \frac{dz}{dt\sqrt{1-u^2/c^2}}; \frac{icdt}{dt\sqrt{1-u^2/c^2}}\right)$$

But $\frac{dx}{dt} = u_x$ etc. ($\vec{u}$ is particle velocity). Hence

$$U = \left(\frac{u_x}{\sqrt{1-u^2/c^2}}; \frac{u_y}{\sqrt{1-u^2/c^2}}; \frac{u_z}{\sqrt{1-u^2/c^2}}; \frac{icdt}{dt\sqrt{1-u^2/c^2}}\right)$$

$$= \left(\frac{u_x}{\sqrt{1-u^2/c^2}}; \frac{u_y}{\sqrt{1-u^2/c^2}}; \frac{u_z}{\sqrt{1-u^2/c^2}}; \frac{ic}{\sqrt{1-u^2/c^2}}\right)$$

$$= \left(\frac{\vec{u}}{\sqrt{1-u^2/c^2}}; \frac{ic}{\sqrt{1-u^2/c^2}}\right) \quad \text{... (1)}$$

The four-vector U is called four-velocity or Minkowski velocity.

The square of the length of the four-velocity is

$$U^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2$$

$$= \frac{u_x^2 + u_y^2 + u_z^2}{1-u^2/c^2} + \frac{i^2 c^2}{1-u^2/c^2}$$

$$= \frac{u^2 - c^2}{1-u^2/c^2} = \frac{-c^2(1-u^2/c^2)}{1-u^2/c^2} = -c^2 \quad \text{... (2)}$$

which is negative.

The four-velocity is then said to be a time like vector.

(C) Momentum Four Vector

If we multiply the four-velocity of a particle by its rest mass (which is an invariant) we get the four-vector called momentum four vector or four-momentum P.

$$P = m_o U = \left(\frac{m_o \vec{u}}{\sqrt{1-u^2/c^2}}, \frac{i m_o c}{\sqrt{1-u^2/c^2}} \right)$$

$$= (\vec{p}, imc) \qquad \text{... (3)}$$

That is $\qquad P_\mu = m_o U_\mu (\mu = 1, 2, 3, 4)$

Note that P is the four-momentum and $\vec{p} = \left(\frac{m_o \vec{u}}{\sqrt{1-u^2/c^2}} \right)$ is the three dimensional momentum vector.

Since $m = E/c^2$, we can write the above equation $P = (\vec{p}, imc)$ as $P = \left(\vec{p}, \frac{iE}{c} \right)$.

Since the length of a four-vector is invariant we must have $p^2 + \left(\frac{iE}{c} \right)^2$ = constant.

$$\therefore \qquad p^2 - \frac{E^2}{c^2} = \text{constant.}$$

When $p = 0$, $E = m_o c^2$

$$\therefore \qquad \text{Constant} = -\frac{(m_o c^2)^2}{c^2} = -m_o^2 c^2$$

$$\therefore \qquad p^2 - \frac{E^2}{c^2} = -m_o^2 c^2$$

or $\qquad E^2 = p^2 c^2 + m_o^2 c^4$

By definition of the four-vector, components of the four-, momentum P in frame S' are:

$$P_1' = \Gamma \left(P_1 + \frac{iv}{c} P_4 \right) \text{ or}$$

$$p_x' = \Gamma \left(p_x + \frac{iv}{c} \frac{iE}{c} \right) = \Gamma \left(p_x - \frac{vE}{c^2} \right) \qquad \text{... (4a)}$$

$$P_2' = P_2, \text{ that is } p_y' = p_y \qquad \text{... (4b)}$$

$$P_3' = P_3, \text{ that is } p_z' = p_z \qquad \text{... (4c)}$$

and $\qquad P_4' = \Gamma \left(P_4 - \frac{iv}{c} P_1 \right)$

or $\qquad \frac{iE'}{c} = \Gamma \left(\frac{iE}{c} - \frac{iv}{c} p_x \right)$

or $\qquad E' = \Gamma (E - v p_x) \qquad \text{... (4d)}$

This equation tells us that what is called as energy in frame S' is a sort of admixture of energy and momentum in frame S.

We can write the inverse transformations as usual by changing the sign of v and interchanging the primed and unprimed quantities. Thus

$$p_x = \Gamma\left(p_x' + \frac{vE'}{c^2}\right);\ p_y = p_y';\ p_z = p_z' \quad \text{and} \quad E = \Gamma(E' + v p_x') \qquad \text{... (5)}$$

These equations of transformation show that momentum and energy and inextricably mixed. For this reason the momentum four-vector is sometimes called momentum-energy four-vector. In Newtonian mechanics we have the principle of conservation of energy and the principle of conservation of momentum. Both these principles are embodied in a single principle of conservation of momentum-energy four-vector in the theory of relativity. We have $P^2 = P'^2$!

For a system of particles $\sum_i P_i = \sum_i P_i'$ where P_i and P_i' are the four-momenta of a particle before and after the collision. Note that this equation in component form gives

$$\Sigma p_x = \Sigma p_x';\ \Sigma p_y = \Sigma p_y';\ \Sigma p_z = \Sigma p_z';\ \Sigma \frac{iE}{c} = \Sigma \frac{iE'}{c} \text{ or } \Sigma E = \Sigma E' \qquad \text{... (6)}$$

(D) Four-Force (Minkowski Force)

Since $\qquad P_\mu = m_o U_\mu (\mu = 1, 2, 3, 4)$

$dP_\mu = m_o dU_\mu$ is a four-vector. Multiplying this four-vector by the scalar $\frac{1}{d\tau}$, we get the four-vector

$$F_\mu = \frac{dP_\mu}{d\tau} = \frac{d}{d\tau}(m_o U_\mu) = m_o \frac{d^2 X_\mu}{d\tau^2}$$

This four-vector F is called the force-vector or Minkowski force.

The equation $F = \frac{dP}{d\tau}$ or $F_\mu = \frac{dP_\mu}{d\tau}$ is a generalization of Newton's equation of motion viz.

$$\vec{f} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{u})$$

The components of the force-vector are

(a) $F_i = \frac{dP_i}{d\tau}$ $(i = 1, 2, 3 \text{ or } x, y, z)$

and (b) $F_4 = \frac{dP_4}{d\tau}$

Consider the first three components of F. They are

$$F_i = \frac{dP_i}{d\tau} = \frac{d}{dt\sqrt{1-u^2/c^2}}\left(\frac{m_o u_i}{\sqrt{1-u^2/c^2}}\right) \qquad \therefore\ d\tau = dt\sqrt{1-u^2/c^2}$$

Or $\qquad F_i\sqrt{1-u^2/c^2} = \frac{d}{dt}\left(\frac{m_o u_i}{\sqrt{1-u^2/c^2}}\right) = \frac{dp_i}{d\tau} = f_i$

$\therefore\ F_i\sqrt{1-u^2/c^2} = f_i = i$th component of ordinary force $\vec{f}$. $\qquad$ ... (6)

In practice it is not convenient to deal with the four-force. It is preferable to use equation (6) in the form

$$\vec{f} = \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right) \qquad \text{... (7)}$$

This is the relativistic equation of motion.

The fourth component of the four-force is

$$F_4 = \frac{dP_4}{d\tau} = \frac{d}{d\tau}\left(\frac{iE}{c}\right) \qquad \because \quad P_4 = iE/c$$

$$= \frac{i}{c}\frac{dE}{d\tau}$$

$$\therefore \qquad F_4 = \left(\frac{i}{c\sqrt{1-u^2/c^2}}\left(\frac{dE}{dt}\right)\right) = \left(\frac{i}{c\sqrt{1-u^2/c^2}}\left(\frac{d(T+m_0c^2)}{dt}\right)\right)$$

where T is the kinetic energy of the particle.

$$\therefore \qquad F_4 = \left(\frac{i}{c}\frac{1}{\sqrt{1-u^2/c^2}}\left(\frac{dT}{dt}\right)\right) \text{ because } m_0 \text{ is an invariant} \qquad \text{... (8)}$$

$\frac{dT}{dt}$ = Rate of change of kinetic energy = $\vec{f} \cdot \vec{u}$ = rate at which force is doing work on the particle.

Thus fourth component of Minkowski force may be written as $F_4 = \left(\frac{i}{c\sqrt{1-u^2/c^2}}\vec{f}\cdot\vec{u}\right)$.

The fourth component of the Minkowski force is seen to be related to the time rate of change of energy or to the rate at which force is doing work on the particle.

Summarizing the four-force is given by

$$F = \left(\frac{\vec{f}}{\sqrt{1-u^2/c^2}}, \frac{i}{c\sqrt{1-u^2/c^2}}\vec{f}\cdot\vec{u}\right) \qquad \text{... (9)}$$

8.4 INTERVAL BETWEEN TWO EVENTS

Let $X_1(x_1, y_1, z_1, ict_1)$ and $X_2(x_2, y_2, z_2, ict_2)$ be the position vectors of two events say A and B respectively in an inertial frame S. Their difference $\Delta X = X_2 - X_1$ is a four vector called the displacement four vector given by

$$\Delta X = [(x_2 - x_1), (y_2 - y_1), (z_2 - z_1); ic(t_2 - t_1)]$$

Its product with itself is an invariant given by

$$I = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = s^2 - c^2t^2$$

where s is the spatial interval or the spatial separation between the two events and t is the temporal separation or time interval between them. In the literature, I is often called the interval between the two events.

In a frame s' moving relative to frame s, the spatial and temporal separations are different ($s' \neq s$; $t' \neq t$). However I remains the same.

If $I > 0$, the interval is spacelike, because this happens when the two events occur at the same time ($t = 0$) and are separated only spatially. That is when $I > 0$, it is always possible to find an inertial system in which the two events occur simultaneously i.e. at the same time.

If $I <, 0$, the interval is time like, because this happens when the two events occur at the same place ($s = 0$) and are separated only in time. Thus when $I < 0$, in any frame, it is always possible to find an inertial frame in which the two events occur at the same place.

If $I = 0$, the interval is light like because this happens only when the two events are connected by a signal travelling at the speed of light i.e. $s = ct$.

SUMMARY

Ordinary three dimensional space plus the time can be considered as making up a four dimensional continuum or Minkowski space. Instead of the usual variables x, y, z and t in frame S, it is convenient to introduce the quantities X_μ ($\mu = 1, 2, 3, 4$) given by $X_1 = x$; $X_2 = y$; $X_3 = z$, $X_4 = ict$. The corresponding quantities in frame s' are

$X'_\mu(\mu = 1, 2, 3, 4)$ with $X'_1 = x'$; $X'_2 = y'$; $X'_3 = z'$; $X'_4 = ict'$.

From L.T. it follows that

$$X'_1 = \Gamma\left(X_1 - \frac{vX_4}{ic}\right) = \Gamma\left(X_1 + \frac{iv}{c}X_4\right)$$

$$X'_2 = X_2;\ X'_3 = X_3 \text{ and } X'_4 = \Gamma\left(X_4 - \frac{iv}{c}X_1\right)$$

Conversely

$$X_1 = \Gamma\left(X_1' - \frac{iv}{c}X_4'\right);\ X_2 = X_2';\ X_3 = X_3' \text{ and } X_4 = \Gamma\left(X_4' + \frac{iv}{c}X_1'\right)$$

A four-vector A_μ is a set of four quantities (A_1, A_2, A_3, A_4) which transform under a L.T. in the same way as the coordinates X_1, X_2 , X_3 , X_4 of a point in the four dimensional space-time continuum. The first three components (A_1, A_2, A_3) are the components of an ordinary three dimensional vector $\vec{A}$ i.e. $A = (\vec{A}, A_4)$.

The sum of two four-vectors is also a four-vector so that $C = A + B$, that is

$$C_\mu = (A + B)_\mu = A_\mu + B_\mu$$

The scaler product of two four-vectors A and $B = \sum_\mu A_\mu B_\mu = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$

It is a scalar quantity.

The square of the length of a four-vector A is defined as the scaler product of the four-vector with itself.

The square of 'length' of a four-vector A is $(\text{length})^2 = \sum_\mu A_\mu A_\mu = A_1^2 + A_2^2 + A_3^2 + A_4^2$

Length of a four-vector is invariant.

If all the four components of a four-vector are multiplied by a scaler (α), we get another four-vector of length α times the original length.

Some important four-vectors introduced in this chapter are:

(i) Position four-vector of a particle X with components $X_1 = x$; $X_2 = y$; $X_3 = z$ and $X_4 = ict$ or $X = (\vec{r}\,,\ ict)$

(ii) Velocity four-vector of a particle U

$$U = \frac{d}{d\tau} \text{(Position four-vector } X\text{) with}$$

Components U_1, U_2, U_3, U_4

$$U = \left(\frac{\vec{u}}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}} \right)$$

(iii) Momentum four-vector of a particle $P = m_0U$.

$$P = \left(\frac{m_0\,\vec{u}}{\sqrt{1-u^2/c^2}}, \frac{im_0\,c}{\sqrt{1-u^2/c^2}} \right)$$

For a system of particles $\sum_i P_i = \sum_i P_i'$ where P_i and P_i' are the four-momenta of a particle before and after the collision.

(iv) Four-force $F = \frac{dP_\mu}{d\tau}$. Its components are (a) $F_i\sqrt{1-u^2/c^2} = f_i = i$th component of ordinary force $\vec{f}\,(i = 1, 2, 3 \text{ or } x, y, z)$ and (b) $F_4 = \frac{i}{c\sqrt{1-u^2/c^2}}\left(\frac{dE}{dt}\right)$

where $\quad E = T + m_0c^2$ = Total particle energy.

$$F = \left(\frac{\vec{f}}{\sqrt{1-u^2/c^2}}, \frac{i}{c\sqrt{1-u^2/c^2}}\,\vec{f}\cdot\vec{u} \right)$$

ILLUSTRATIVE EXAMPLES

Example 1 Given two four-vectors $A = (\vec{i}\,a_1 + \vec{j}\,a_2 + \vec{k}\,a_3, 5ic)$ and $B = (\vec{i}\,b_1 + \vec{j}\,b_2 + \vec{k}\,b_3, -4ic)$ find (a) their sum $C = A + B$ and (ii) their scaler product AB.

Solution

(a) $\quad C_\mu = (A + B)_\mu = A_\mu + B_\mu$

$\therefore\ C = \lfloor \vec{i}(a_1 + b_1) + \vec{j}(a_2 + b_2) + \vec{k}(a_3 + b_3), (5 - 4)ic \rfloor$

Thus $C_1 = (a_1 + b_1)$; $C_2 = (a_2 + b_2)$; $C_3 = (a_3 + b_3)$ and $C_4 = ic$

(b) Their sealer product $= \sum_\mu A_\mu B_\mu$

$$= a_1b_1 + a_2b_2 + a_3b_3 + 20c^2.$$

Example 2 Show that four-momentum and four-force are mutually perpendicular, that is show that the sealer product of P and F is zero. Hence show that the rate of change of energy of a particle is equal to the rate at which the force does work on the particle.

Solution

$$P.P = \sum_{\mu} P_{\mu}^2 = m_0^2 \sum_{\mu} U_{\mu}^2 = -m_0^2 c^2 = \text{constant} \quad \text{... (1)}$$

since $\sum_{\mu} U_{\mu}^2 = -c^2$

Differentiating eqn. (1) with respect to τ we get

$$P \cdot \frac{dP}{d\tau} = P.F. = 0 \quad \text{... (2)}$$

Thus the scalar product of P and F is zero.
From Eqn. (2) we see that $P_1F_1 + P_2F_2 + P_3F_3 + P_4F_4 = 0$

$$\therefore \quad P_4F_4 = -(P_1F_1 + P_2F_2 + P_3F_3) = -\vec{p} \cdot \frac{\vec{f}}{\sqrt{1-u^2/c^2}}$$

$$\therefore \quad \left(\frac{im_0 c}{\sqrt{1-u^2/c^2}}\right)\left(\frac{i}{c\sqrt{1-u^2/c^2}} \frac{dE}{dt}\right) = -\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right) \cdot \frac{\vec{f}}{\sqrt{1-u^2/c^2}}$$

$$\therefore \quad \frac{dE}{dt} = \vec{u} \cdot \vec{f}$$

Hence the rate of change of energy of a particle is equal to the rate at which the force does work on the particle.

Example 3 Pions of sufficiently high energy on colliding with stationary protons, lead to the reaction $\pi + p \rightarrow K + \Sigma$. What is the minimum total pion energy required to bring about this reaction? What is the pion kinetic energy then?

Solution Let the four-momentum of the system before collision in the laboratory frame when the pion carries minimum total energy required be $P = \left[p, \frac{i}{c}(E_{\pi}^0 + m_p c^2) \right]$ where p is the pion momentum.

Let P' be the four-momentum of the system before the collision and Q the four-momentum of the system after the reaction in the c. of m. frame.

Then $\quad Q = [0, \ i(m_k + m_{\Sigma})c]$

By conservation of four-momentum $P' = Q$ and by invariance of length of four momentum, $P^2 = P'^2$

$$\therefore \quad P^2 = Q^2$$

$$\therefore \quad p^2 - \frac{1}{c^2}(E_{\pi}^{02} + m_p^2 c^4 + 2E_{\pi}^0 m_p c^2) = -(m_k + m_{\Sigma})^2 c^2$$

Substituting $p^2 = \dfrac{E_\pi^{02} - m_\pi^2 c^4}{c^2}$ we get

$$\frac{E_\pi^{02} - m_\pi^2 c^4}{c^2} - \frac{1}{c^2}(E_\pi^{02} + m_p^2 c^4 + 2E_\pi^0 m_p c^2) = -(m_k + m_\Sigma)^2 c^2$$

$$\therefore \qquad 2E_\pi^0 m_p = (m_k + m_\Sigma)^2 c^2 - (m_\pi^2 + m_p^2)c^2$$

$$\therefore \qquad E_\pi^0 = \left[\frac{(m_k + m_\Sigma)^2 - (m_\pi^2 + m_p^2)}{2m_p}\right]c^2$$

This is the total minimum energy required. The minimum pion kinetic energy is

$$E_\pi^0 - m_\pi c^2 = \left[\frac{(m_k + m_\Sigma)^2 - (m_\pi + m_p)^2}{2m_p}\right]c^2$$

Example 4 A gamma ray photon of energy E is incident on a stationary proton to produce pions as per the reaction: p + gamma ray $\to \pi + p$. If the pion is emitted at 90° to the direction of the incident photon, show that its energy is

$$E_\pi = \left[\frac{(E.m_p + m_\pi^2 c^2/2}{E + m_p c^2}\right]c^2$$

Solution

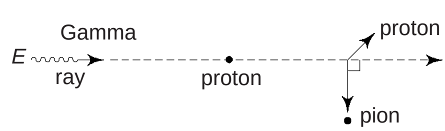

Fig. 8.1 For Illustrative Example 4

Let $P = \left[\frac{E}{c}\vec{a_1}, i\left(\frac{E}{c} + m_p c\right)\right]$ be the four-momentum of the system before the reaction where $\vec{a_1}$ is a unit vector in the direction of the incident gamma ray photon. Let $Q = Q_\pi + Q_p = \left(p_\pi \vec{a_2}, i\frac{E_\pi}{c}\right)$ $+ \left(p\vec{a_3}, i\frac{E_p}{c}\right)$ be the four-momentum of the system after the reaction. Note that $\vec{a_2}$ is a unit vector in the direction of the pion produced and $\vec{a_3}$ is a unit vector in the direction of the outgoing proton of momentum p.

Since $\qquad P = Q = Q_\pi + Q_p$; $Q_p = P - Q_\pi$ or

$$Q_P^2 = P^2 + Q_\pi^2 - 2P\cdot Q_\pi$$

$$\therefore \quad \left(p\vec{a_3}, \frac{iE_p}{c}\right)^2 = \left[\frac{E}{c}\vec{a_1}, i\left(\frac{E}{c} + m_p c\right)\right]^2 + \left(p_\pi \vec{a_2}, i\frac{E_\pi}{c}\right)^2 - 2\left[\frac{E}{c}\vec{a_1}, i\left(\frac{E}{c} + m_p c\right)\right]\cdot\left(p_\pi \vec{a_2}, i\frac{E_\pi}{c}\right)$$

$$\therefore\ (p^2 - E_p^2/c^2) = \frac{E^2}{c^2} - \frac{E^2}{c^2} - m_p^2 c^2 - 2Em_p + \left(p_\pi^2 - \frac{E_\pi^2}{c^2}\right) - 2\left[0 - \left(\frac{E}{c} + m_p c\right)E_\pi/c\right]$$

because $\vec{a}_1 \cdot \vec{a}_2 = 0$. See Fig 8.1 where the pion travels at right angles to the direction of the incident gamma ray photon. Multiplying the above equation by c^2, using $p^2c^2 - E^2 = -m^2c^4$ for the proton and the pion and simplifying we get the required result viz.

$$E_\pi = \left[\frac{(Em_p + m_\pi c^2/2)}{E + m_p c^2}\right] c^2 \text{ as required.}$$

Example 5 A photon of energy E collides with a stationary electron of rest mass m_0. As a result of the collision, the photon energy is reduced to $E' < E$ and the photon is observed to be deflected through an angle θ. Using appropriate four-vectors, show that the increase in photon wavelength is given by

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta).$$

Solution

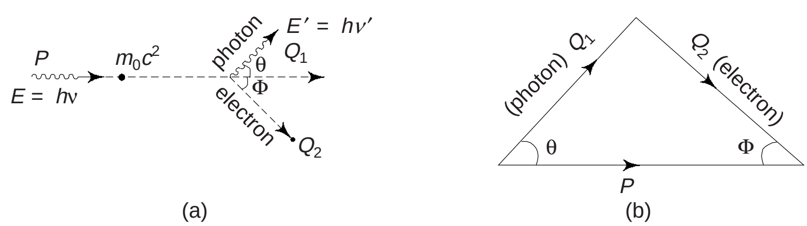

Fig. 8.2 For Illustrative Example 5

Let $P = \left[\frac{E}{c}\vec{a}_1, i\left(\frac{E + m_0c^2}{c}\right)\right]$ be the four-momentum of the system before collision. Let $Q_1 = \left[\frac{E'}{c}\vec{a}_2, \frac{iE'}{c}\right]$ be the four-momentum of the photon deflected through angle θ. Note that $\vec{a}_1$ is a unit vector in the direction of the incident photon and $\vec{a}_2$ is a unit vector in the direction of the photon deflected through angle θ, i.e. $\vec{a}_1 \cdot \vec{a}_2 = \cos\theta$. Let $Q_2 = \left[p_e\vec{a}_3, \frac{iE_e}{c}\right]$ be the four-momentum of the recoiling electron. Then $P = Q_1 + Q_2$. This is shown in Fig. 8.2(b). Since $Q_2 = P - Q_1$

$$Q_2^2 = P^2 + Q_1^2 - 2PQ_1$$

$$\therefore\ \left(p_e\vec{a}_3, \frac{iE_e}{c}\right)^2 = \left[\frac{E}{c}\vec{a}_1, i\left(\frac{E + m_o c^2}{c}\right)\right]^2 + \left(\frac{E'}{c}\vec{a}_2, \frac{iE'}{c}\right)^2 - 2\left[\frac{E}{c}\vec{a}_1, i\left(\frac{E + m_o c^2}{c}\right)\right]\cdot\left(\frac{E'}{c}\vec{a}_2, \frac{iE'}{c}\right)$$

$$\therefore \quad \left(p_e^2 - \frac{E_e^2}{c^2}\right) = \frac{E^2}{c^2} - \left(\frac{E^2}{c^2} + m_0^2 c^2 + 2Em_0\right) + \left(\frac{E'^2}{c^2} - \frac{E'^2}{c^2}\right) + \frac{2EE'}{c^2} + 2m_0 E' - \frac{2EE'}{c^2}\cos\theta$$

Multiplying throughout by c^2 and using

$$(p_e^2 c^2 - E_e^2) = -m_0^2 c^4 \text{ we get}$$

$$\therefore \quad -m_0^2 c^4 = -m_0^2 c^4 - 2Em_0c^2 - 2EE' \cos\theta + 2EE' + 2m_0c^2E'$$

$$\therefore \quad EE'(1 - \cos\theta) = m_0c^2(E - E')$$

$$\therefore \quad E - E' = \frac{EE'(1-\cos\theta)}{m_0 c^2}$$

Let $E = \frac{hc}{\lambda}$ and $E' = \frac{hc}{\lambda'}$; λ and λ' being the wavelengths of the incident and scattered photon respectively. Then

$$hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h^2 c^2 (1-\cos\theta)}{\lambda\lambda'(m_0 c^2)}$$

$$\therefore \quad \frac{\lambda' - \lambda}{\lambda\lambda'} = \frac{h}{m_0 c}\frac{(1-\cos\theta)}{\lambda\lambda'}$$

$\therefore$ Increase in wavelength $\lambda' - \lambda = \Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta)$

EXERCISES

1. What is space-time continuum? Write down the L.T.s for the variables X_μ (μ = 1, 2, 3, 4) in frame S. How are the inverse relations obtained?
2. What are four-vectors? Explain how two four-vectors are added and multiplied.
3. Show that the ‘length’ of a four-vector in unchanged by a L.T.
4. Define four-velocity and four-momentum for a particle. Show that they are time like.
5. Define four-acceleration as $A = \frac{dU}{d\tau}$. Show that four-velocity and four-acceleration are orthogonal.
6. Let four-force $F = \frac{dP}{d\tau}$. Show that the four force can be completely expressed in terms of the ordinary force vector $\vec{f}$ as

   $$F = (1 - u^2/c^2)^{-1/2}\left(\vec{f}\,;\frac{i}{c}\vec{f}\cdot\vec{u}\right)$$

7. Show that the four-velocity is of constant magnitude ic.
8. Show that $P = \left(\vec{p}, \frac{iE}{c}\right)$ and deduce that $p^2 - \frac{E^2}{c^2}$ is an invariant $-m_0^2c^2$ with respect to a L.T.

9. A gamma ray of energy E incident on a stationary proton produces a pion as per the reaction $p + \gamma \rightarrow \pi + p$.

(a) Show that the threshold energy required for the reaction is $E_0 = m_\pi c^2 \left[1 + \frac{m_\pi}{2m_p}\right]$

(b) If the pion is emitted at right angles to the incident gamma ray photon, show that its energy is

$$E_\pi = \left[\frac{(Em_p + m_\pi^2 c^2/2)}{E + m_p c^2}\right] c^2$$

10. Show that the threshold in kinetic energy at which an accelerated proton beam will begin to produce proton antiproton pair (using stationary proton target) in the reaction $p + p \rightarrow p + p + p + p^-$, is $6m_pc^2$ where m_p is the rest mass of proton/antiproton.

11. A stationary nucleus of mass M is in an excited state of excitation energy E_0. It emits a photon in a transition which takes it to the ground state. Show that the frequency of the photon is

$$\nu = \frac{E_0}{h}\left(1 - \frac{E_0}{2Mc^2}\right)$$

Hint: rest mass of the atom in the ground state is $M_o = \left(M - \frac{E_0}{c^2}\right)$ and its momentum is $\frac{E}{c} = \frac{h\nu}{c}$.

12. A beam of negatively charged pions of energy E strikes a target of stationary protons to give rise to the reaction $\pi^- + p \rightarrow n + \gamma$

If the neutron emerges at 90° to the direction of the incident pion beam, show that its energy is

$$E_n = m_pc^2 + \frac{m_\pi^2 c^4}{2(E + m_p c^2)}$$

The difference between neutron and proton mass is negligible.

Suggested Further Reading

Rindler W.: "*Special Relativity*" (Wiley Interscience)
Rosser W.G.V.: *An Introduction to The Theory of Relativity*. (Butterworth London).

Chapter

9

RELATIVITY AND ELECTROMAGNETISM

INTRODUCTION

While relativity brings about basic changes in our concepts about mass, momentum etc. in mechanics, it is not so in case of electricity and magnetism studied before the advent of relativity.

The laws of electricity and magnetism are enshrined in a set of equations known as Maxwell's Equations. These equations are invariant under L.T. Historically L.T. were derived by assuming that M.E. are invariant in going from one frame to another.

In fact relativity allows a more coherent study of electricity and magnetism or rather of electromagnetism. Indeed we find that magnetism is a relativistic phenomenon because given Coulomb's law in electrostatics, magnetism can be 'discovered' in going from one inertial frame to another. A field that is purely electric or purely magnetic in one frame of reference is found to have both electric and magnetic components in another frame. In this chapter we shall have a glimpse of relativistic electrodynamics; that is application of special relativity to electromagnetism.

9.1 CHARGE DENSITY AND CURRENT DENSITY

Imagine a cube of volume l_0^3 containing a charge Q. The cube is stationary in frame S' shown in Fig. 9.1. The charge per unit volume i.e. the charge density in frame S' is $\rho_0 = Q/l_0^3$. The charge Q is at

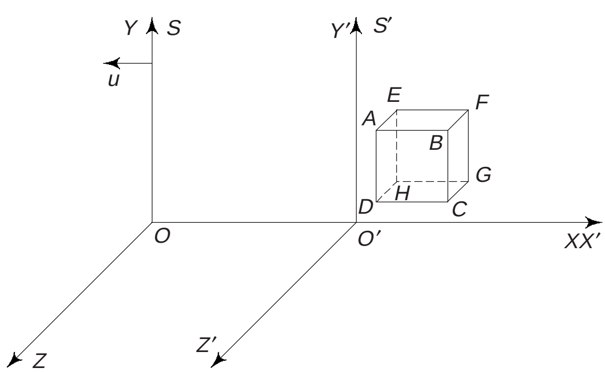

Fig. 9.1 Charge Density and Current Density

rest in frame S' and hence there is no electric current. In particular, the current density or the current per unit cross-sectional area in frame S' is $j_0 = 0$.

Now consider the situation from frame S which is moving with velocity u parallel to the edge CD of the cube. The length of the edge CD is now contracted to $l = l_0\sqrt{1-u^2/c^2}$ but the transverse edges of the cube still measure l_o. Hence the volume of the cube in frame S is $l_0^3\sqrt{1-u^2/c^2}$. The charge density is

$$\rho = \frac{Q}{l_0^3\sqrt{1-u^2/c^2}} = \frac{\rho_0}{\sqrt{1-u^2/c^2}} \quad \text{... (1)}$$

In frame S, the charge moves to the right with velocity u and the current density = (charge density) (velocity of charge). Hence

$$j = \rho u = \frac{\rho_0\, u}{\sqrt{1-u^2/c^2}} \quad \text{... (2)}$$

In a general case if we consider a current density $\vec{j}$ with components j_x, j_y and j_z, we would get

$$j_x = \frac{\rho_0 u_x}{\sqrt{1-u^2/c^2}},\ j_y = \frac{\rho_0 u_y}{\sqrt{1-u^2/c^2}},\ j_z = \frac{\rho_0 u_z}{\sqrt{1-u^2/c^2}} \quad \text{... (3)}$$

and $\rho = \dfrac{\rho_0}{\sqrt{1-u^2/c^2}}$ (1) as above.

In fact if we multiply the numerator and the denominator by m_o in equations (1) and (3), they may be concisely written as

$$\rho = \frac{\rho_0 m}{m_0} \quad \text{and} \quad \vec{j} = \frac{\rho_0}{m_0}\vec{p} \quad \text{... (4)}$$

where m_o is the mass of a particle at rest in frame S' and m and $\vec{p}$ are the mass and momentum respectively of that same particle in frame S.

The quantities $\vec{j}$ and ρ transform just like momentum $\vec{p}$ and mass m ($m = E/c^2$): (See Art. 6.9). Without further ado, we may therefore write the equations of transformation for current and charge density as (in our usual convention)

$$j_x' = \frac{j_x - \rho v}{\sqrt{1-v^2/c^2}};\ j_y' = j_y;\ j_z' = j_z \quad \text{... (5a)}$$

$$\rho' = \frac{\rho - v j_x/c^2}{\sqrt{1-v^2/c^2}} \quad \text{... (5b)}$$

and their inverses

$$j_x = \frac{j_x' + \rho' v}{\sqrt{1-v^2/c^2}};\ j_y = j_y';\ j_z = j_z' \quad \text{... (6a)}$$

$$\rho = \frac{\rho' + vj_x'/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{... (6b)}$$

when frame S' moves with velocity v relative to frame S along X-X' axis as usual.

9.2 INTERDEPENDENCE OF ELECTRIC AND MAGNETIC FIELDS

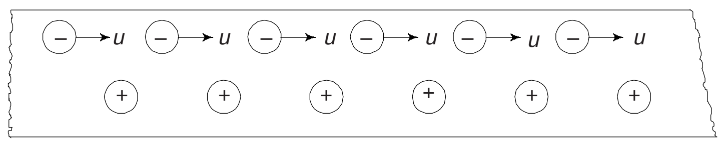

Fig. 9.2 Interdependence of electric magnetic fields

Consider a simplified model of a current carrying wire. Imagine a long straight wire (in frame S) in which an electric current flows due to free electrons moving with a drift velocity u to the right. Let n denote the number density (number per unit volume) of free electrons as well as that of positive ions in the wire. The separation between electrons is the same as that between the positive ions so that the net charge in any volume of the wire is zero. The positive ions are at rest in frame S and the electric current is constituted only by the motion of electrons.

In frame S, there is a negative charge density $\rho^- = -ne$ due to electrons and a positive charge density $\rho^+ = +ne$ due to positive ions. The net charge density is then $\rho = \rho^- + \rho^+ = 0$.

In frame S, there is a current density $j_x^- = -neu = \rho^- u$ due to electrons and $j_x^+ = 0$ due to positive ions. The net current density is $j_x = j_x^- + j_x^+ = \rho^- u$.

Consider next the current carrying wire as seen in frame S' moving to the right along X-X' axis relative to frame S. An observer in frame S' not only finds the positive charges to be moving to the left with velocity v but also that the current carrying wire (neutral in S) is electrically charged as shown below.

From equation (4a), the charge density in S' is

$$\rho'^- = \frac{\rho^- - vj_x^-/c^2}{\sqrt{1 - v^2/c^2}} \text{ and } \rho'^+ = \frac{\rho^+ - vj_x^+/c^2}{\sqrt{1 - v^2/c^2}}$$

Since $j_x^- = \rho^- u$ and $j_x^+ = 0$, we get

$$\rho'^- = \rho^- \frac{(1 - vu/c^2)}{\sqrt{1 - v^2/c^2}} = \Gamma\rho^-(1 - vu/c^2).$$

$$\rho'^+ = \frac{\rho^+}{\sqrt{1 - v^2/c^2}} = \Gamma\rho^+.$$

Substituting $\rho^- = -ne$ and $\rho^+ = +ne$, the net charge density is given by

$$\rho' = \rho'^+ + \rho'^- = \Gamma ne[1 - (1 - vu/c^2)] = \Gamma nevu/c^2.$$

The net charge density as observed by an observer in frame S' is therefore positive. In other words an observer in frame S' concludes that the current carrying wire is positively charged.

We now see that according to the observer in S, the net charge density is zero and he does not observe any electric field. But since there is a net current in the wire, he observes a magnetic field around the wire. In frame S, there is only a magnetic field B but no electric field. On the other hand, an observer in S'

observes a positively charged wire which is also carrying a current. Hence in frame S', there is not only some magnetic field, B' (due to current) but also some electric field E' (due to positive charge). Thus what is observed as a pure magnetic field by one observer is observed as a combination of electric and magnetic fields by another observer. In other words whether an electromagnetic field is purely electric or purely magnetic is not absolute but depends on the inertial frame from which sources of electromagnetic fields are observed. This is the interdependence of electric and magnetic fields.

9.3 TRANSFORMATION OF ELECTRIC AND MAGNETIC FIELDS

A. Electric Fields

We know that the force (called Lorentz force) on a charged particle in an electromagnetic field is given by $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$ where q is the charge of the particle, $\vec{u}$ its velocity and $\vec{E}$ and $\vec{B}$ are respectively the electric and magnetic fields at the location of the particle.

Let S and S' denote two frames of reference with frame S' moving to the right along X-X' axis with velocity $\vec{v}$ as usual. Let $\vec{E}'$ and $\vec{B}'$ denote respectively the electric and magnetic fields in frame S' in which a charge q is stationary. Since the charge is stationary, it does not experience any magnetic force in frame S' and the electric force on it, is given by $\vec{F}' = q\vec{E}'$ with components $F_x' = qE_x'$; $F_y' = qE_y'$ and $F_z' = qE_z'$.

In frame S the charge has velocity $\vec{v}$ (the velocity of S' relative to S) and the electromagnetic force on it, is given by $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Components of $\vec{v}$ are $v_x = v$, $v_y = v_z = 0$.

From the transformation of force, we know that

$$F_x = F_x'; \; F_y = F_y'/\Gamma \text{ and } F_z = F_z'/\Gamma$$

Now the x component of force F is

$$F_x = q[E_x + (\vec{v} \times \vec{B})_x] = q[E_x + v_yB_z - v_zB_y] = qE_x$$
$$= F_x' = qE_x'$$

∴ $$E_x' = E_x \qquad \text{... (1a)}$$

Similarly $$F_y = q[E_y + (\vec{v} \times \vec{B})_y] = q[E_y + v_zB_x - v_xB_z]$$
$$= q[E_y - vB_z] = F_y'/\Gamma = qE_y'/\Gamma$$

∴ $$E_y' = \Gamma(E_y - vB_z) \qquad \text{... (1b)}$$

and $$F_z = q[E_z + (\vec{v} \times \vec{B})_z] = q[E_z + v_xB_y - v_yB_x]$$
$$= q(E_z + vB_y] = F_z'/\Gamma = qE_z'/\Gamma$$

∴ $$E_z' = \Gamma(E_z + vB_y) \qquad \text{... (1c)}$$

Inverse relations obtained by changing v to $-v$ and interchanging primed and unprimed quantities are:

$$E_x = E_x'; \; E_y = \Gamma(E_y' + vB_z') \quad \text{and} \quad E_z = \Gamma(E_z' - vB_y') \qquad \text{... (2)}$$

If E_{11} and $E_\perp$ denote the electric field components parallel and perpendicular respectively to the direction of the relative velocity $\vec{v}$, then equations (1) may be written as

$$E'_{11} = E_{11} \text{ and } E_\perp' = (E_\perp + (\vec{v} \times \vec{B})_\perp) \qquad \text{... (3)}$$

as may be easily verified.

Equations (3) show that components of $\vec{E}$ parallel to the relative velocity of the two frames remains unchanged, whereas components transverse to the relative velocity transform into part electric and part magnetic fields.

(B) Magnetic Fields

Let S and S' denote two inertial frames of reference with frame S' moving to the right along X-X' axis with velocity $\vec{v}$ as usual. Let $\vec{E}'$ and $\vec{B}'$ denote respectively the electric and magnetic fields in frame S' and $\vec{E}$ and $\vec{B}$ the corresponding fields in frame S.

Consider first an electric charge q which is moving in frame S' with velocity u' in the Y' direction so that $u_x' = u_z' = 0$ and $u_y' = u'$.

In frame S', the force on charge q is

$$\vec{F}' = q(\vec{E}' + \vec{u}' \times \vec{B}') \text{ with components}$$

$$\vec{F}_x' = q\lfloor E_x' + (\vec{u}' \times \vec{B}')_x \rfloor = q[E_x' + u_y'B_z' - u_z'B_y']$$

$$= q(E_x' + u_y'B_z) \qquad \text{... (4a) because } u_z' = 0.$$

Similarly, $F_y' = q[E_y' + (\vec{u}' \times \vec{B}')_y] = q[E_y' + u_z'B_x' - u_x'B_z']$

$\therefore$ $F_y' = qE_y'$ ... (4b) since $u'_z = u'_x = 0$

and $F_z' = q[E_z' + (\vec{u}' \times \vec{B}')_z] = q[E_z' + u_x'B_y' - u_y'B_x']$

$\therefore$ $F_z' = q[E_z' - u_y'B_x']$ ... (4c) since $u'_x = 0$.

The charge q has velocity $\vec{u}'$ along Y' in frame S'. Since S' is travelling with velocity $\vec{v}$ along X-X' axis, the velocity of charge q in frame S is given by $\vec{u}$ with components.

$$u_x = \frac{u_x' + v}{1 + u_x'v/c^2} = v \qquad \text{... (5a)}$$

$$u_y = \frac{u_y'}{\Gamma(1 + u_x'v/c^2)} = u_y'/\Gamma \qquad \text{... (5b)}$$

and
$$u_z = \frac{u_z'}{\Gamma(1 + u_x'v/c^2)} = 0 \qquad \text{... (5c)}$$

In frame S, the force on charge q is $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$ with components

$$F_x = q[E_x + (\vec{u} \times \vec{B})_x] = q[E_x + u_yB_z - u_zB_y].$$

$\therefore$ $F_x = q(E_x + u_yB_z)$ ... (6a)

since $u_z = 0$.

Similarly $F_y = q[E_y + (\vec{u} \times \vec{B})_y] = q[E_y + u_zB_x - u_xB_z]$.

$\therefore$ $F_y = q(E_y - vB_z)$ ... (6b)

since $u_z = 0$.

and $F_z = q[E_z + (\vec{u} \times \vec{B})_z] = q[E_z + u_xB_y - u_yB_x]$.

$\therefore$ $F_z = q[E_z + vB_y - u_yB_x]$ ... (6c)

Since the charge is in motion in S', for transformation of forces acting on it, we must use the equations (derived previously in chapter 6)

$$F_x = \frac{F_x' + v/c^2(\vec{u}' \cdot \vec{F}')}{1 + u_x'v/c^2} = F_x' + v/c^2\,(u_y' \cdot F_y') \qquad \text{... (7a)}$$

since $u'_x = u'_z = 0.$

$$F_y = \frac{F'_y}{\Gamma(1 + u'_x v/c^2)} = \frac{F'_y}{\Gamma} \quad \text{... (7b)}$$

$$F_z = \frac{F'_z}{\Gamma(1 + u'_x v/c^2)} = \frac{F'_z}{\Gamma} \quad \text{... (7c)}$$

From Eqns. (6a) and (7a) we get

$F_x = q(E_x + u_y B_z) = F'_x + \dfrac{v}{c^2}(u'_y F'_y)$. Using Eqns. (4a) and (4b) we get

$$q(E_x + u_y B_z) = q(E'_x + u'_y B'_z) + \frac{v}{c^2}(u'_y \, qE'_y)$$

Since $E_x = E'_x$ and $u_y = u'_y/\Gamma$, above equation simplifies to

$$\frac{u'_y B_z}{\Gamma} = u'_y B'_z + \frac{v}{c^2}(u'_y E'_y).$$

$$\therefore \quad \frac{B_z}{\Gamma} = B'_z + \frac{v}{c^2} E'_y$$

From Eqn. (1b) $E'_y = \Gamma(E_y - vB_z)$. Hence above eqn. becomes

$$\frac{B_z}{\Gamma} = B'_z + \frac{v}{c^2}\Gamma(E_y - vB_z) = B'_z + \frac{v}{c^2}\Gamma E_y - \frac{v^2}{c^2}\Gamma B_z$$

$$\therefore \quad \Gamma B_z\left(v^2/c^2 + \frac{1}{\Gamma^2}\right) = B'_z + \frac{\Gamma v}{c^2} E_y$$

$$\therefore \quad \Gamma B_z = B'_z + \frac{\Gamma v}{c^2} E_y \qquad \text{since} \qquad \frac{1}{\Gamma^2} = 1 - v^2/c^2$$

$$\therefore \quad B'_z = \Gamma\left(B_z + \frac{v}{c^2} E_y\right) \quad \text{... (8c)}$$

From eqns. (6c) and (7c) we get

$$F_z = q(E_z + vB_y - u_y B_x) = \frac{F'_z}{\Gamma} = q(E'_z - u'_y B'_x)/\Gamma$$

Putting $u_y = u'_y/\Gamma$ and $E'_z = \Gamma(E_z + vB_y)$, we get

$$E_z + vB_y - \frac{u'_y \, B_x}{\Gamma} = \Gamma(E_z + vB_y)/\Gamma - \frac{u'_y \, B'_x}{\Gamma}$$

$$\therefore \quad B'_x = B_x \quad \text{... (8a)}$$

Suppose next that charge q in frame S is moving in the Z' direction (instead of Y' above) with velocity $\vec{u}'$ so that $u'_x = u'_y = 0$ and $u'_z = u'$.

In frame S', the force on charge q is $\vec{F}' = q(\vec{E}' + \vec{u}' \times \vec{B}')$ with components

$$F'_x = q\lfloor E'_x + (\vec{u}' \times \vec{B}')_x \rfloor = q[E'_x + u'_y B'_z - u'_z B'_y]$$
$$= q(E'_x - u'_z B'_y) \quad \text{... (9a)}$$

$$F_y' = q\lfloor E_y' + u_z'B_x' - u_x'B_z' \rfloor = q(E_y' + u_z'B_x') \qquad \text{... (9b)}$$

and $$F_z' = q\lfloor E_z' + u_x'B_y' - u_yB_x' \rfloor = qE_z' \qquad \text{... (9c)}$$

The charge q has a velocity $\vec{u}\,'$ along z' axis in frame S'. In frame S, its velocity $\vec{u}$ now has components given by

$$u_x = \frac{u_x' + v}{1 + u_x' v/c^2} = v \qquad \text{... (10a)}$$

$$u_y = \frac{u_y'}{\Gamma(1 + u_x' v/c^2)} = 0 \qquad \text{... (10b)}$$

and $$u_z = \frac{u_z'}{\Gamma(1 + u_x' v/c^2)} = \frac{u_z'}{\Gamma} \qquad \text{... (10c)}$$

In frame S, the force on charge q is given by $\vec{F} = q\lfloor(\vec{E} + (\vec{u} \times \vec{B})\rfloor$ with x component

$$F_x = q[E_x + (\vec{u} \times \vec{B})_x] = q[E_x + u_yB_z - u_zB_y].$$

$$= q(E_x - u_zB_y) = q\left(E_x - \frac{u_z' B_y}{\Gamma}\right) \text{ since } u_y = 0 \text{ and } u_z = u_z'/\Gamma.$$

Now $$F_x = \frac{F_x' + v/c^2(\vec{u}\,' \cdot \vec{F}\,')}{1 + u_x' v/c^2}.$$ When $u_x' = u_y' = 0$, we get

$$F_x = F_x' + v/c^2\,(u_z' \cdot F'_z)$$

$$\therefore \quad qF_x = q(E_x - u_z'B_y/\Gamma) = q(E_x' - u_z'B_y') + \frac{qv}{c^2} \cdot u_z'E_z'$$

$$\therefore \quad -B_y/\Gamma = -B_y' + \frac{v}{c^2} \cdot E_z'$$

$$\therefore \quad B_y/\Gamma = B_y' - \frac{v}{c^2}\Gamma\,(E_z + vB_y) \text{ because } E_z' = \Gamma(E_z + vB_y)$$

$$\therefore \quad B_y' = \Gamma B_y\left[v^2/c^2 + \frac{1}{\Gamma^2}\right] + \Gamma\frac{v}{c^2} \cdot E_z$$

$$\therefore \quad B_y' = \Gamma\left(B_y + \frac{v}{c^2}E_z\right) \qquad \text{... (8b)}$$

Inverse transformations are easily obtained by changing v to $-v$ and interchanging primed and unprimed quantities. All these equations will be needed in future. For convenience we write them all here for easy reference in future.

$$\left.\begin{aligned} E_x' &= E_x \\ E_y' &= \Gamma(E_y - vB_z) \\ E_z' &= \Gamma(E_z + vB_y) \\ B_x' &= B_x \\ B_y' &= \Gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ B_z' &= \Gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned}\right\} \quad \text{... (11)} \qquad \left.\begin{aligned} E_x &= E_x' \\ E_y &= \Gamma(E_y' + vB_z') \\ E_z &= \Gamma(E_z' - vB_y') \\ B_x &= B_x' \\ B_y &= \Gamma\left(B_y' - \frac{v}{c^2}E_z'\right) \\ B_z &= \Gamma\left(B_z' + \frac{v}{c^2}E_y'\right) \end{aligned}\right\} \quad \text{... (12)}$$

Two special cases are:

(A) There is only an electric field in frame S, so that $\vec{B} = 0$. Then $\vec{B}' = \vec{i}B'_x + \vec{j}B'_y + \vec{k}B'_z$. Using equations (11) repeatedly we get

$$\vec{B}' = \frac{\Gamma v}{c^2}(\vec{j}E_z - \vec{k}E_y) = \frac{v}{c^2}(\vec{j}E'_z - \vec{k}E'_y)$$

Since $\vec{v} = \vec{i}v_x + \vec{j}v_y + \vec{k}v_z = \vec{i}v_x = \vec{i}v$, above equation can be written as

$$\vec{B}' = -\frac{\Gamma}{c^2}(\vec{v} \times \vec{E}) = -\frac{1}{c^2}(\vec{v} \times \vec{E}') \qquad \text{... (13)}$$

Therefore a pure electric field in S is observed to have electric as well as magnetic field components in S'.

(B) There is only a magnetic field in frame S, so that $\vec{E} = 0$.

Then $\vec{E}' = \vec{i}E'_x + \vec{j}E'_y + \vec{k}E'_z$

Using equations (11) repeatedly, we get

$$\vec{E}' = -\Gamma v(\vec{j}B_z - \vec{k}B_y) = -v\,(\vec{j}B'_z - \vec{k}B'_y)$$

Again we can write this as

$$\vec{E}' = \Gamma(\vec{v} \times \vec{B}) = \vec{v} \times \vec{B}' \qquad \text{... (14)}$$

Therefore a pure magnetic field in S is observed to have electric as well as a magnetic field component in frame S'.

Since $B'_x = 0$ in equation (13) and $E'_x = 0$ in equation (14), we can write above equations respectively as

$$\vec{B}' = \vec{B}'_{\perp} = -\frac{\Gamma}{c^2}(\vec{v} \times \vec{E}) = -\frac{1}{c^2}(\vec{v} \times \vec{E}')$$

and $$\vec{E}' = \vec{E}'_{\perp} = \Gamma(\vec{v} \times \vec{B}) = \vec{v} \times \vec{B}'$$

9.4 ELECTRIC AND MAGNETIC FIELDS OF A UNIFORMLY MOVING CHARGE

Figure 9.3(a) shows a source charge q moving with constant velocity v relative to frame S along the X-axis. At the time $t = 0$, the charge is instantaneously located at the origin of S. To determine the electric and magnetic fields produced by this charge q in frame S, we first find the fields produced by the charge q in frame S'; which is moving with velocity v relative to frame S as shown in Fig. 9.3(a). The charge q is stationary in S'; and is located at its origin O'.

The electric field at a point $P(x', y', z')$ in S' due to the stationary charge q at the origin O' is

$$\vec{E}' = \frac{q\vec{r}'}{4\pi\epsilon_0 r'^3}$$ with components.

$$E'_x = \frac{qx'}{4\pi\epsilon_0 r'^3};\; E'_y = \frac{qy'}{4\pi\epsilon_0 r'^3} \text{ and } E'_z = \frac{qz'}{4\pi\epsilon_0 r'^3} \qquad \text{... (1)}$$

where $\overrightarrow{O'P} = \vec{r}'$.

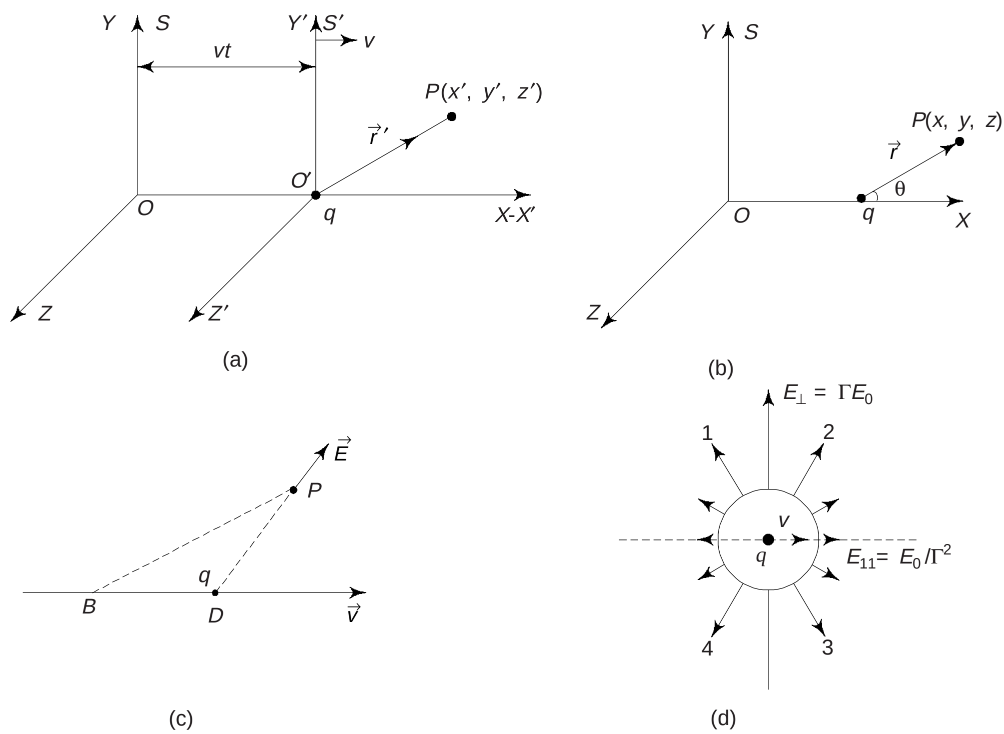

Fig. 9.3 Electric and Magnetic Fields of a Uniformly moving charge

Since the charge q is stationary in S', it does not produce any magnetic field and $\vec{B}' = 0$.

(A) Electric Field

Let x, y, z denote the coordinates of point P in frame S. In this frame the charge q is located at ($x = vt$, 0, 0) at time t. The electric field E (at time t) can be obtained from the transformation equations which in the present case are given by $E_x = E_x'$; $E_y = \Gamma E_y'$ and $E_z = \Gamma E_z'$ since $B_x' = B_y' = B_z' = 0$

$$\therefore \qquad E_x = E_x' = \frac{qx'}{4\pi\epsilon_0 r'^3} \qquad \text{... (2a)}$$

$$E_y = \Gamma E_y' = \frac{\Gamma q y'}{4\pi\epsilon_0 r'^3} \qquad \text{... (2b)}$$

and

$$E_z = \Gamma E_z' = \frac{\Gamma q z'}{4\pi\epsilon_0 r'^3} \qquad \text{... (2c)}$$

To express these field components in terms of x, y, z and t, we use L.T. viz.

$$x' = \Gamma(x - vt), \; y' = y \text{ and } z' = z.$$

$$\therefore \qquad r' = (x'^2 + y'^2 + z'^2)^{1/2} = [\Gamma^2(x - vt)^2 + y^2 + z^2]^{1/2}$$

Above field components can be now written as

$$E_x = \frac{q\Gamma(x - vt)}{4\pi\epsilon_0[\Gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}} \quad \text{... (3a)}$$

$$E_y = \frac{q\Gamma y}{4\pi\epsilon_0[\Gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}} \quad \text{... (3b)}$$

$$E_z = \frac{q\Gamma z}{4\pi\epsilon_0[\Gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}} \quad \text{... (3c)}$$

Let $\vec{r} = \vec{i}(x - vt) + \vec{j}y + \vec{k}z$ denote the vector drawn from the charge q to point P making angle θ with X-axis. See Fig. 9.3(b). Then $(x - vt) = r\cos\theta$ and $(y^2 + z^2) = r^2\sin^2\theta$.

$\therefore \quad \Gamma^2(x - vt)^2 + y^2 + z^2 = \Gamma^2 r^2\cos^2\theta + r^2\sin^2\theta$

$$= \Gamma^2 r^2\left(\cos^2\theta + \frac{\sin^2\theta}{\Gamma^2}\right) = \Gamma^2 r^2\left(1 - \sin^2\theta + \frac{\sin^2\theta}{\Gamma^2}\right)$$

$$= \Gamma^2 r^2\left[1 - \sin^2\theta\left(1 - \frac{1}{\Gamma^2}\right)\right] = \Gamma^2 r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)$$

Electric field $\vec{E}$ in frame S can now be written as

$$\vec{E} = \frac{q\Gamma\vec{r}}{4\pi\epsilon_0\Gamma^3 r^3\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}$$

$$\therefore \quad \vec{E} = \frac{q(1 - \beta^2)}{4\pi\epsilon_0 r^2(1 - \beta^2\sin^2\theta)^{3/2}}\frac{\vec{r}}{r} \quad \text{... (4)}$$

where $\quad \frac{1}{\Gamma^2} = \left(1 - \frac{v^2}{c^2}\right) = 1 - \beta^2.$

If the charge q were stationary in frame S, the electric field in S would be given by

$$\vec{E}_0 = \frac{q}{4\pi\epsilon_0 r^2}\frac{\vec{r}}{r}$$

Hence, above equation can also be written as

$$\vec{E} = \frac{\vec{E}_0(1 - \beta^2)}{(1 - \beta^2\sin^2\theta)^{3/2}} \quad \text{... (5)}$$

Some noteworthy points about the electric field are given below:

(a) when $\theta = 0$ or $180°$, $\sin\theta = 0$ and above equation can be written as

$$\vec{E} = \vec{E}_{11} = \vec{E}_0(1 - \beta^2) = \vec{E}_0/\Gamma^2$$

Thus as compared with the usual field due to stationary charge, in the present case the field is decreased by a factor of $1/\Gamma^2$ at a point in line ahead or behind the moving (source) charge.

(b) In the direction transverse to the motion of the charge $\sin\theta = \pm 1$ and $\vec{E} = \vec{E}_\perp = \dfrac{\vec{E}_0}{(1-\beta^2)^{1/2}} = \Gamma \vec{E}_0$

Thus as compared with the usual field due to a stationary charge, in the present case, the field is increased by a factor Γ in the direction transverse to the motion of the charge. Obviously the field varies continuously from one value to another in intermediate directions. This is shown in Fig. 9.3(d).

(c) A surprising aspect of the equation (4) or (5) is that at any instant of time in frame S, the electric field at a point is along the line joining the instantaneous position of the charge to the point at which the field is to be determined. See Fig. 9.3(c). At the instant the charge q is at D, the electric field $\vec{E}$ at point P is along DP as shown.

If we suppose that any effect or signal emanating from the charge can travel at the most with speed c, then we expect that the charge should have been at its earlier position B, so that in some definite time interval Δt, while the signal travels the distance BP at speed c, the charge starting from B arrives at D by travelling at speed v.

Then $$\Delta T = \frac{\text{distance covered}}{\text{speed}} = \frac{BP}{c} = \frac{BD}{v}$$

or $$v/c = \frac{BD}{BP}$$

We therefore expect the field to originate from the earlier position B of the charge rather than its present position D. That the field seems to originate from point D rather than point B is definitely surprising but true!

(B) Magnetic Field

Components of magnetic field are given by

$$B_x = B_x';\ B_y = \Gamma\left(B_y' - \frac{v}{c^2}E_z'\right) \text{ and } B_z = \Gamma\left(B_z' + \frac{v}{c^2}E_y'\right)$$

Since $B_x' = B_y' = B_z' = 0$, they reduce to

$$B_x = 0;\ B_y = \Gamma\left(\frac{-v}{c^2}E_z'\right) \text{ and } B_z = \Gamma\left(\frac{v}{c^2}E_y'\right)$$

Since $E_y = \Gamma E_y'$ and $E_z = \Gamma E_z'$, the magnetic field components in frame S are given by $B_x = 0$,

$$B_y = -\frac{v}{c^2}E_z \text{ and } B_z = \frac{v}{c^2}E_y \qquad \text{... (6)}$$

or $$B_{11} = 0 \text{ and } B_\perp = \frac{\vec{v}\times\vec{E}}{c^2} \qquad \text{... (7)}$$

Since $v_x = v$, $v_y = v_z = 0$, we may write the above two equations as a single one:

$$\vec{B} = \frac{\vec{v}\times\vec{E}}{c^2} \qquad \text{... (8)}$$

9.5 MAGNETISM AS A RELATIVISTIC PHENOMENON

Magnetism is often introduced in an ad hoc manner as if magnetic forces are totally distinct from electric forces. Our intention here is to show that magnetism is a relativistic phenomenon, that is when relativity is applied to Coulomb's law of electrostatics, magnetic forces emerge as a natural consequence of going from one inertial frame to another inertial frame.

In fact magnetism can be 'discovered' by applying transformation of forces to Coulomb force for stationary electric charges.

Consider the two inertial frames S and S' shown in Fig. 9.4. Frame S' moves to the right with velocity v as shown. Charges Q and q are a distance r apart and stationary in frame S'. By Coulomb's law, they repel each other and tend to move apart along Y' axis. In particular, the force exerted on charge Q is

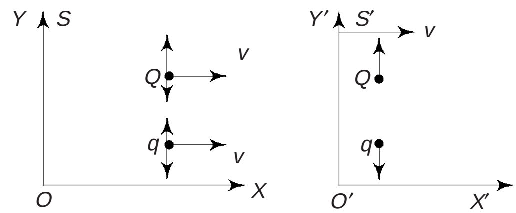

Fig. 9.4 Magnetism as a relativistic phenomenon

$$\vec{F}' = \vec{j} F_y = \frac{\vec{j}}{4\pi\epsilon_0} \frac{qQ}{r^2} \quad \text{... (1)}$$

The force on charge q is $-\vec{F}'$.

When observed from frame S', the two charges are still a distance r apart. However, now they are observed to have a velocity $\vec{v}$ along X-axis as shown.

In frame S, the force experienced by the charge Q can be found by making use of equations of transformation of force which in the present case are given by $F_x = F_x'$; $F_y = F_y'/\Gamma$ and $F_z = F_z'/\Gamma$.

Since $F_x' = F_z' = 0$, the net force acting on the upper charge is $\vec{j} F_y$ where

$$F_y = F_y'/\Gamma = \left(\frac{qQ}{4\pi\epsilon_0 r^2}\right)\frac{1}{\Gamma} \quad \text{... (2a)}$$

$$= \left(\frac{qQ}{4\pi\epsilon_0 r^2}\right)(1 - v^2/c^2)^{1/2} \quad \text{... (2b)}$$

which is less than the static force of repulsion $\left(\frac{qQ}{4\pi\epsilon_0 r^2}\right)$.

We attribute this to a 'new force', the magnetic force of attraction which reduces the Coulomb force of repulsion.

$F_y = F'_y$ = normal Coulomb force only when $v = 0$. Thus magnetic force makes its appearance only when $v \neq 0$ or when charges are in motion. This is the reason why magnetic force is not present in frame S' where the charges are stationary i.e. when there is no relative motion between the observer and the charges.

Magnetism is thus a relativistic phenomenon. It is in effect a second order relativistic effect because it depends upon $(v/c)^2$, the square of charge velocity to the velocity of light.

9.6 INVARIANCE OF MAXWELL'S EQUATIONS

The simplest way to establish invariance of Maxwell's equations would be to show that a spherical light wave spreads uniformly with velocity c in all inertial frames. The student may easily do this by using L.T. to show that $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$. We shall proceed here in a more formal way.

The student is assumed to be familiar with Maxwell's equations which in the usual notation are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{... (1a)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{... (1b)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{... (1c)}$$

and

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \qquad \text{... (1d)}$$

The simplest of these containing only the fields $\vec{E}$ and $\vec{B}$ viz. (1b) and (1c) written in component forms are

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad \text{... (2)}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \qquad \text{... (3)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \qquad \text{... (4)}$$

and

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \qquad \text{... (5)}$$

The L.T. are $x' = \Gamma(x - vt)$; $y' = y$, $z' = z$ and $t' = \Gamma(t - vx/c^2)$

Also

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'} + \frac{\partial t'}{\partial x}\frac{\partial x}{\partial t'} \qquad \text{... (6)}$$

$\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ and $\frac{\partial}{\partial t}$ can be obtained similarly by replacing x by y, z and t respectively.

From L.T.;

$$\frac{\partial x'}{\partial x} = \Gamma; \ \frac{\partial y'}{\partial x} = \frac{\partial z'}{\partial x} = 0 \text{ and } \frac{\partial t'}{\partial x} = -\Gamma v/c^2 \qquad \text{... (7)}$$

$\therefore$

$$\frac{\partial}{\partial x} = \Gamma \left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'} \right) \qquad \text{... (8a)}$$

Similarly

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \qquad \text{... (8b)}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \qquad \text{... (8c)}$$

and $$\frac{\partial}{\partial t} = \Gamma\left(\frac{\partial}{\partial t'} - \frac{v\partial}{\partial x'}\right) \quad \text{... (8d)}$$

Consider first equation (3) viz. $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$

Substituting $\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}; \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$ and $\frac{\partial}{\partial t} = \Gamma\left(\frac{\partial}{\partial t'} - \frac{v\partial}{\partial x'}\right)$

we get $$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = -\Gamma\left(\frac{\partial}{\partial t'} - \frac{v\partial}{\partial x'}\right)B_x.$$

$$\therefore \quad \frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = -\frac{\partial \Gamma B_x}{\partial t'} + \frac{\Gamma v \partial B_x}{\partial x'}$$

Substituting $E_z = \Gamma(E_z' - vB_y')$, $E_y = \Gamma(E_y' + vB_z')$ and $B_x = B_x'$, we get

$$\Gamma\left[\frac{\partial}{\partial y'}(E_z' - vB_y') - \frac{\partial}{\partial z'}(E_y' + vB_z')\right] = -\Gamma\left(\frac{\partial B_x'}{\partial t'} - v\frac{\partial B_x'}{\partial x'}\right)$$

$$\therefore \quad \frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} = -\frac{\partial B_x'}{\partial t'} + v\left(\frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'}\right)$$

or $$\frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = -\frac{1}{v}\left(\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} + \frac{\partial B_x'}{\partial t'}\right) \quad \text{... (9)}$$

Consider next equation (2) viz. $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$

Substituting as above we get

$$\Gamma\left(\frac{\partial B_x}{\partial x'} - \frac{v}{c^2}\frac{\partial B_x}{\partial t'}\right) + \frac{\partial B_y}{\partial y'} + \frac{\partial B_z}{\partial z'} = 0$$

Substituting $B_x = B_x'$; $B_y = \Gamma\left(B_y' - \frac{v}{c^2}E_z'\right)$ and $B_z = \Gamma\left(B_z' + \frac{v}{c^2}E_y'\right)$

we get $$\Gamma\left[\left(\frac{\partial B_x'}{\partial x'} + \frac{v}{c^2}\frac{\partial B_x'}{\partial t'}\right) + \frac{\partial B_y'}{\partial y'} - \frac{v}{c^2}\frac{\partial E_z'}{\partial y'} + \frac{\partial B_z'}{\partial z'} + \frac{v}{c^2}\frac{\partial E_y'}{\partial z'}\right] = 0$$

$$\therefore \quad \left(\frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'}\right) = \frac{v}{c^2}\left(\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} + \frac{\partial B_x'}{\partial t'}\right) \quad \text{... (10)}$$

From equations (9) and (10) we see that

$$\frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = 0 \quad \text{and} \quad \frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} + \frac{\partial B_x'}{\partial t'} = 0$$

$\therefore$ $$\nabla \cdot \vec{B}' = 0 \qquad \text{... (11)}$$

and $$(\nabla \times \vec{E}')_x = -\left(\frac{\partial \vec{B}'}{\partial t'}\right)_x \qquad \text{... (12)}$$

Consider now equation (4) viz. $\dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = -\dfrac{\partial B_y}{\partial t}$

Proceeding as above, we get

$$\frac{\partial E_x}{\partial z'} - \Gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)(E_z) = -\Gamma\left(\frac{\partial}{\partial t'} - \frac{v\partial}{\partial x'}\right)B_y$$

$\therefore$ $$\frac{\partial E_x}{\partial z'} - \frac{\partial}{\partial x'}\,[\Gamma(E_z + vB_y)] = -\frac{\partial}{\partial t'}\left[\Gamma\left(B_y + \frac{v}{c^2}E_z\right)\right]$$

From the equations of transformation $E_x = E_x'$

$$E_z' = \Gamma\,(E_z + vB_y) \quad \text{and} \quad B_y' = \Gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$\therefore$ $$\frac{\partial E_x'}{\partial z'} - \frac{\partial E_z'}{\partial x'} = -\frac{\partial}{\partial t'}B_y$$

$\therefore$ $$(\nabla \times \vec{E}')_y = -\left(\frac{\partial \vec{B}'}{\partial t'}\right)_y \qquad \text{... (13)}$$

Consider next, equation $\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = -\dfrac{\partial B_z}{\partial t}$

We now get $\Gamma\left(\dfrac{\partial}{\partial x'} - \dfrac{v}{c^2}\dfrac{\partial}{\partial t'}\right)E_y - \dfrac{\partial E_x}{\partial y'} = -\Gamma\left(\dfrac{\partial}{\partial t'} - \dfrac{v\partial}{\partial x'}\right)B_z$

$\therefore$ $$\frac{\partial}{\partial x'}\,[\Gamma\,(E_y - vB_z)] - \frac{\partial E_x}{\partial y'} = -\frac{\partial}{\partial t'}\left[\Gamma\left(B_z - \frac{v}{c^2}E_y\right)\right]$$

From the equations of transformation $E_x = E_x'$

$$E_y' = \Gamma(E_y - vB_z) \text{ and } B_z' = \Gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$\therefore \qquad \frac{\partial E_y'}{\partial x'} - \frac{\partial E_x'}{\partial y'} = -\frac{\partial}{\partial t'} B_z'$$

$$\therefore \qquad (\nabla \times \vec{E}')_z = -\left(\frac{\partial \vec{B}'}{\partial t'}\right)_z \qquad \text{... (14)}$$

From equations (11), (12), (13) and (14) we see that

$$\nabla \cdot \vec{B}' = 0 \qquad \text{... (15)}$$

and

$$\nabla \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t'} \qquad \text{... (16)}$$

Remaining Maxwell's equations are

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \text{ and } \nabla \times \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Consider first the equation

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}$$

Since $\rho = \Gamma\left(\rho' + \frac{v}{c^2} j_x'\right)$, on substituting for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, we get

$$\Gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)E_x + \frac{\partial E_y}{\partial y'} + \frac{\partial E_z}{\partial z'} = \frac{\Gamma}{\varepsilon_0}\left(\rho' + \frac{v}{c^2} j_x'\right)$$

$$\therefore \quad \Gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right) E'_x + \frac{\partial}{\partial y'} \Gamma(E'_y + vB'_y) + \frac{\partial}{\partial z'} \Gamma(E'_z - vB'_y) = \frac{\Gamma}{\varepsilon_0}\left(\rho' + \frac{v}{c^2} j_x'\right)$$

$$\frac{\partial E_x'}{\partial x'} + \frac{\partial E_y'}{\partial y'} + \frac{\partial E_z'}{\partial z'} - \frac{\rho'}{\varepsilon_0} = \frac{v}{c^2}\left(\frac{\partial E_x'}{\partial t'} - j_x'/\varepsilon_0\right) + v\left(\frac{\partial B_y'}{\partial z'} - \frac{\partial B_z'}{\partial y'}\right)$$

Using $c^2 = 1/\mu_0\varepsilon_0$ and rearranging we get

$$\frac{\partial E_x'}{\partial x'} + \frac{\partial E_y'}{\partial y'} + \frac{\partial E_z'}{\partial z'} - \frac{\rho'}{\varepsilon_0} = -v\left(\frac{\partial B_z'}{\partial y'} - \frac{\partial B_y'}{\partial z'} - \mu_0\, \varepsilon_0 \frac{\partial E_x'}{\partial t'} - \mu_0\, j_x'\right) \qquad \text{... (17)}$$

From the last M.E. $(\nabla \times \vec{B})_x = \mu_0 \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)_x$

$$\therefore \qquad \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 \left(j_x + \varepsilon_0 \frac{\partial E_x}{\partial t} \right)$$

since $j_x = \Gamma(j'_x + \rho' v)$, we get

$$\frac{\partial}{\partial y'}\left[\Gamma\left(B'_z + \frac{v}{c^2}E'_y\right)\right] - \frac{\partial}{\partial z'}\left[\Gamma\left(B'_y - \frac{v}{c^2}E'_z\right)\right] = \mu_0\Gamma(j'_{x'} + \rho' v) + \mu_0\,\varepsilon_0\Gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)E'_x$$

$$\therefore \quad \frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'} - \mu_0\left(j'_x + \varepsilon_0\frac{\partial E'_x}{\partial t'}\right)$$

$$= -\frac{v}{c^2}\left(\frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} - \mu_0 c^2 \rho'\right)$$

$$= -\frac{v}{c^2}\left(\frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} - \frac{\rho'}{\varepsilon_0}\right) \qquad \text{... (18)}$$

From equations (17) and (18) we see that

$$(\nabla \times \vec{B}')_x = \mu_0\left(\vec{j}' + \varepsilon_0\frac{\partial \vec{E}'}{\partial t'}\right)_x \qquad \text{... (19)}$$

and

$$\nabla \cdot \vec{E}' = \rho'/\varepsilon_0 \qquad \text{... (20)}$$

Consider next $(\nabla \times \vec{B})_y = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)_y$

Hence $\dfrac{\partial B_x}{\partial z} - \dfrac{\partial B_z}{\partial x} = \mu_0\left(j_y + \varepsilon_0\dfrac{\partial E_y}{\partial t}\right)$

$$\text{L.H.S.} = \frac{\partial B_x}{\partial z'} - \Gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)B_z$$

$$= \frac{\partial B_x}{\partial z'} - \frac{\Gamma\partial B_z}{\partial x'} + \frac{\Gamma v}{c^2}\frac{\partial B_z}{\partial t'}$$

$$= \frac{\partial B'_x}{\partial z'} - \frac{\Gamma^2\partial}{\partial x'}\left(B'_z + \frac{v}{c^2}\right)E'_y + \frac{\Gamma^2 v}{c^2}\frac{\partial}{\partial t'}\left(B'_z + \frac{v}{c^2}E'_y\right)$$

$$= \frac{\partial B'_x}{\partial z'} - \frac{\Gamma^2\partial B'_z}{\partial x'} - \frac{\Gamma^2 v}{c^2}\frac{\partial E'_y}{\partial x'} + \frac{\Gamma^2 v}{c^2}\frac{\partial B'_z}{\partial t'} + \frac{\Gamma^2 v^2}{c^4}\frac{\partial E'_y}{\partial t'}$$

$$\text{R.H.S.} = \mu_0 j'_y + \mu_0\varepsilon_0\Gamma\left(\frac{\partial}{\partial t'} - \frac{v\partial}{\partial x'}\right)E_y$$

$$= \mu_0 j'_y + \mu_0\varepsilon_0\Gamma^2 \frac{\partial}{\partial t'}(E'_y + vB'_z) - \mu_0\varepsilon_0\Gamma^2 \frac{v\partial}{\partial x'}(E'_y + vB'_z)$$

$$= \mu_0 j'_y + \mu_0\varepsilon_0\Gamma^2 \frac{\partial E'_y}{\partial t'} + \mu_0\varepsilon_0\Gamma^2 v\frac{\partial B'_z}{\partial t'} - \mu_0\varepsilon_0\Gamma^2 v\frac{\partial E'_y}{\partial x'} - \mu_0\varepsilon_0\Gamma^2 v^2\frac{\partial B'_z}{\partial x'}$$

On inspection we find that coefficients of $\frac{\partial B'_z}{\partial t'}$ and $\frac{\partial E'_y}{\partial x'}$ are the same on either side. The two sides can now be combined to give.

$$\frac{\partial B'_x}{\partial z'} - \frac{\partial B'_z}{\partial x'}(\Gamma^2 - \Gamma^2 v^2/c^2) + \frac{\partial E'_y}{\partial t'}\left(\frac{\Gamma^2 v^2}{c^4} - \frac{\Gamma^2}{c^2}\right) = \mu_0 j'_y$$

$$\therefore \quad \frac{\partial B'_x}{\partial z'} - \frac{\partial B'_z}{\partial x'}\Gamma^2(1 - v^2/c^2) = \mu_0 j'_y + \frac{\Gamma^2}{c^2}(1 - v^2/c^2)\frac{\partial E'_y}{\partial t'}$$

$$\therefore \quad \frac{\partial B'_x}{\partial z'} - \frac{\partial B'_z}{\partial x'} = \mu_0 j'_y + \mu_0\varepsilon_0\frac{\partial E'_y}{\partial t'} \qquad \because c^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$\therefore \quad (\nabla \times \vec{B}')_y = \mu_0\left(\vec{j}' + \varepsilon_0\frac{\partial \vec{E}'}{\partial t}\right)_y \qquad \text{... (21)}$$

Finally consider $(\nabla \times \vec{B})_z = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)_z$

Then $\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0\left(j_z + \varepsilon_0\frac{\partial E_z}{\partial t}\right)$

$$\text{L.H.S.} = \Gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)B_y - \frac{\partial B_x}{\partial y'} = \frac{\Gamma\partial B_y}{\partial x'} - \frac{\Gamma v}{c^2}\frac{\partial B_y}{\partial t'} - \frac{\partial B_x}{\partial y'}$$

$$= \Gamma^2\frac{\partial}{\partial x'}\left(B'_y - \frac{v}{c^2}E'_z\right) - \frac{\Gamma^2 v}{c^2}\frac{\partial}{\partial t'}\left(B'_y - \frac{v}{c^2}E'_z\right) - \frac{\partial B'_x}{\partial y'}$$

$$= \Gamma^2\frac{\partial B'_y}{\partial x'} - \frac{\Gamma^2 v}{c^2}\frac{\partial E'_z}{\partial x'} - \frac{\Gamma^2 v}{c^2}\frac{\partial B'_y}{\partial t'} + \frac{\Gamma^2 v^2}{c^4}\frac{\partial E'_z}{\partial t'} - \frac{\partial B'_x}{\partial y'}$$

$$\text{R.H.S.} = \mu_0 j_z + \mu_0\varepsilon_0\frac{\partial E_z}{\partial t} = \mu_0 j_z + \mu_0\varepsilon_0\Gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)E_z.$$

$$= \mu_0 j_z + \mu_0\varepsilon_0\Gamma^2 \frac{\partial}{\partial t'}(E_z' - vB_y') - \mu_0\varepsilon_0\Gamma^2 v \frac{\partial}{\partial x'}(E_z' - vB_y')$$

$$= \mu_0 j_z' + \mu_0\varepsilon_0\Gamma^2 \frac{\partial E_z'}{\partial t'} - \mu_0\varepsilon_0\Gamma^2 v \frac{\partial B_y'}{\partial t'} - \mu_0\varepsilon_0\Gamma^2 v \frac{\partial E_z'}{\partial x'} + \mu_0\varepsilon_0\Gamma^2 v^2 \frac{\partial B_y'}{\partial x'}$$

Canceling common quantities on either side, we get

$$\frac{\partial B_y' \,\Gamma^2}{\partial x'}\left(1 - \frac{v^2}{c^2}\right) - \frac{\partial B_x'}{\partial y'} = \mu_0 j_z' + \frac{\partial E_z'}{\partial t'}\frac{\Gamma^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore \qquad \frac{\partial B_y'}{\partial x'} - \frac{\partial B_x'}{\partial y'} = \mu_0 j'_z + \mu_0\varepsilon_0 \frac{\partial E_z'}{\partial t'} \text{ since } c^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$\therefore \qquad (\nabla \times \vec{B}')_z = \mu_0 \left(\vec{j}' + \varepsilon_0 \frac{\partial \vec{E}'}{\partial t'}\right)_z \qquad \text{... (22)}$$

From equations (12), (16), (20) and equations (19), (21) and (22) taken together we have shown that

$$\nabla \cdot \vec{E}' = \rho'/\varepsilon_\circ$$

$$\nabla \cdot \vec{B}' = 0$$

$$\nabla \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t'}$$

and

$$\nabla \times \vec{B}' = \mu_0 \left(\vec{j}' + \varepsilon_0 \frac{\partial \vec{E}'}{\partial t'}\right)$$

Thus Maxwell's equations are invariant.

SUMMARY

Charge density is the charge per unit volume. Current density is the current per unit crossectional area. Their equations of transformation are

$$j'_x = \Gamma(j_x - \rho v); \qquad j'_y = j_y; \qquad j'_z = j_z$$

$$\rho' = \Gamma(\rho - vj_x/c^2)$$

Their inverses are easily written down.

What is observed as a pure electric (or pure magnetic) field by one observer, is observed as a combination of electric and magnetic fields by another observer. In other words whether an electromagnetic field is purely electric or purely magnetic is not absolute but depends on the inertial frame in which the field is observed.

The equations of transformation of electric and magnetic fields are

$$E'_x = E_x; \qquad E'_y = \Gamma(E_y - vB_z); \qquad E'_z = \Gamma(E_z + vB_y)$$

$$B'_x = B_x; \qquad B'_y = \Gamma\left(B_y + \frac{v}{c^2}E_z\right); \qquad B'_z = \Gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

Their inverses can be written down by changing the sign of v and interchanging the primed and unprimed quantities.

Electric and magnetic fields produced by a uniformally moving charge are given by

$$E = \frac{q\,(1-\beta^2)}{4\pi\varepsilon_o r^2\,(1-\beta^2\sin^2\theta)^{3/2}}\,\frac{\vec{r}}{r}$$

and
$$\vec{B} = \frac{\vec{v}\times\vec{E}}{c^2}$$

When relativity is applied to Coulomb's law in electrostatics, magnetic forces appear merely as a consequence of going from one inertial frame to another. Magnetism is therefore a relativistic phenomenon.

Maxwell's equations in electromagnetism viz.

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0; \qquad \nabla\cdot\vec{B} = 0; \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ and}$$

$$\nabla\times\vec{B} = \mu_0\left(\vec{j}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \text{ are found to be invariant.}$$

ILLUSTRATIVE EXAMPLES

Example 1 Show that $c^2\rho^2 - (j_x^2 + j_y^2 + j_z^2) = c^2\rho_0^2$

Solution We know that

$$\rho = \frac{\rho_0}{\sqrt{1-u^2/c^2}};\ j_x = \frac{\rho_0 u_x}{\sqrt{1-u^2/c^2}};\ j_y = \frac{\rho_0 u_y}{\sqrt{1-u^2/c^2}} \text{ and } j_z = \frac{\rho_0 u_z}{\sqrt{1-u^2/c^2}}$$

$$\therefore\ c^2\rho^2 - (j_x^2 + j_y^2 + j_z^2) = \frac{c^2\rho_0^2}{(1-u^2/c^2)} - \frac{\rho_0^2}{(1-u^2/c^2)}(u_x^2+u_y^2+u_z^2)$$

$$= \frac{c^2\rho_0^2}{1-u^2/c^2} - \frac{\rho_0^2u^2}{1-u^2/c^2} = \frac{\rho_0^2}{1-u^2/c^2}(c^2-u^2)$$

$$= \frac{\rho_0^2c^2\,(1-u^2/c^2)}{1-u^2/c^2} = \rho_0^2c^2$$ which is an invariant since ρ_0 is the charge density in a frame in which the charge is at rest.

Example 2 Show that $E^2 - c^2B^2$ is invariant under a L.T.

Solution We know that under a L.T., $\vec{E}$ and $\vec{B}$ transform as follows:

$$E_x = E_x'; \; E_y = \Gamma(E_y' + vB_z'); \; E_z = (E_z' - vB_y')$$

$$B_x = B_x'; \; B_y = \Gamma\left(B_y' - \frac{v}{c^2}E_z'\right); \; B_z = \Gamma\left(B_z' + \frac{v}{c^2}E_y'\right)$$

$$\therefore \quad E^2 - c^2B^2 = E_x^2 + E_y^2 + E_z^2 - c^2(B_x^2 + B_y^2 + B_z^2)$$

$$= E_x'^2 + \Gamma^2(E_y'^2 + 2E_y' vB_z' + v^2 B_z'^2) + \Gamma^2(E_z'^2 - 2E_z' vB_y' + v^2 B_y'^2) - c^2 B_x'^2$$

$$- \Gamma^2 c^2 \left(B_y'^2 - 2B_y' \frac{v}{c^2}E_z' + \frac{v^2}{c^4}E_z'^2 + B_z'^2 + 2B_z' \frac{v}{c^2}E_y' + \frac{v^2}{c^4}E_y'^2\right)$$

$$= E_x'^2 - c^2 B_x'^2 + \Gamma^2 E_y'^2\left(1 - \frac{v^2}{c^2}\right) + \Gamma^2 E_z'^2\left(1 - \frac{v^2}{c^2}\right) + \Gamma^2 v^2 B_y'^2\left(1 - \frac{c^2}{v^2}\right) + \Gamma^2 v^2 B_z'^2\left(1 - \frac{c^2}{v^2}\right)$$

$$= E_x'^2 - c^2 B_x'^2 + E_y'^2 + E_z'^2 + \Gamma^2 v^2 B_y'^2\left(\frac{v^2 - c^2}{v^2}\right) + \Gamma^2 v^2 B_z'^2\left(\frac{v^2 - c^2}{v^2}\right)$$

$$= (E_x'^2 + E_y'^2 + E_z'^2) - c^2 B_x'^2 + \Gamma^2 c^2\left(\frac{v^2}{c^2} - 1\right)B_y'^2 + \Gamma^2 c^2\left(\frac{v^2}{c^2} - 1\right)B_z'^2$$

$$= (E_x'^2 + E_y'^2 + E_z'^2) - c^2(B_x'^2 + B_y'^2 + B_z'^2) \qquad \therefore \; \Gamma^2 = 1 - v^2/c^2$$

$$= E'^2 - c^2B'^2.$$

Example 3 Given that $(E^2 - c^2B^2)$ and $(\vec{E} \cdot \vec{B})$ are invariant under a L.T., show that for any given electromagnetic field, it is possible to find an inertial frame in which either $\vec{E} = 0$ (if $E < cB$) or $\vec{B} = 0$ (if $E > cB$) at a given point in the electromagnetic field if and only if $\vec{E} \cdot \vec{B} = 0$ at that point.

Solution Let $\vec{E}'$ and $\vec{B}'$ denote the electric and magnetic fields respectively in a frame S'. For any other frame S with corresponding fields $\vec{E}$ and $\vec{B}$, the fields must fulfill the conditions.

$$E'^2 - c^2B'^2 = E^2 - c^2B^2 \text{ and } \vec{E}' \cdot \vec{B}' = \vec{E} \cdot \vec{B}$$

If $E' < cB'$ or $E'^2 - c^2B'^2$ is negative, it is possible to find a frame S in which $\vec{E} = 0$ because then $E < cB$ or $E^2 - c^2B^2$ is negative as required. However, when $\vec{E} = 0$, the fields in frame S' are

$$E_x' = E_x = 0 \qquad B_x' = B_x$$

$$E_y' = \Gamma(E_y - vB_z) = -\Gamma vB_z \qquad B_y' = \Gamma\left(B_y + \frac{v}{c^2}E_z\right) = \Gamma B_y$$

$$E_z' = \Gamma(E_z + vB_y) = \Gamma vB_y \qquad B_z' = \Gamma\left(B_z - \frac{v}{c^2}E_y\right) = B_z\Gamma$$

Then $\vec{E}' \cdot \vec{B}' = E_x' B_x' + E_y' B_y' + E_z' B_z' = (-\Gamma v B_z)\Gamma B_y + \Gamma v B_y(\Gamma B_z) = 0.$

Hence it is possible to find the frame S with $\vec{E} = 0$ if and only if $\vec{E} \cdot \vec{B} = 0$.

On the other hand when $E' > cB'$ or $E'^2 - c^2B'^2$ is positive in frame S', it is possible to find a frame S in which $\vec{B} = 0$, because then $E > cB$ or $E^2 - c^2B^2$ is positive as required. However, when $\vec{B} = 0$, the fields in frame S' are

$$E_x' = E_x \qquad B_x' = B_x = 0$$

$$E_y' = \Gamma(E_y - vB_z) = \Gamma E_y \qquad B_y' = \Gamma\left(B_y + \frac{v}{c^2}E_z\right) = \frac{\Gamma v}{c^2}E_z$$

$$E_z' = \Gamma(E_z + vB_y) = \Gamma E_z \qquad B_z' = \Gamma\left(B_z - \frac{v}{c^2}E_y\right) = -\frac{\Gamma v}{c^2}E_y$$

Then $\vec{E}' \cdot \vec{B}' = E_x' B_x' + E_y' B_y' + E_z' B_z' = (\Gamma E_y)\left(\frac{\Gamma v}{c^2}E_z\right) + (\Gamma E_z)\left(-\frac{\Gamma v}{c^2}E_y\right) = 0$

Hence it is possible to find the frame S in which $\vec{B} = 0$ if and only if $\vec{E}.\vec{B} = 0$.

Example 4 A charge q at rest in frame S' experiences a force $\vec{F}' = q\vec{E}'$. If the frame S' is moving with velocity v to the right along X-X' axis, use the equations of force and field transformations to show that as observed in frame S, the force is given by the Lorentz force viz.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Solution In frame S', the force $\vec{F}'$ has components $F_x' = qE_x'$; $F_y' = qE_y'$ and $F_z' = qE_z'$.

In frame S, the force $\vec{F}$ experienced by the charge has components

$$F_x = F_x' = qE_x'; \; F_y = F_y'/\Gamma = \frac{qE_y'}{\Gamma} \text{ and } F_z = F_z'/\Gamma = \frac{qE_z'}{\Gamma}$$

From the transformations of field components

$$F_x = qE_x' = qE_x$$

$$F_y = qE_y'/\Gamma = \frac{q\Gamma}{\Gamma}(E_y - vB_z) = qE_y - qvB_z$$

$$F_z = qE_z'/\Gamma = \frac{q\Gamma}{\Gamma}(E_z + vB_y) = qE_z + qvB_y$$

If $\vec{E}$ is the electric field in frame S, the electric force is

$$\vec{F_e} = q(\vec{i}E_x + \vec{j}E_y + \vec{k}E_z) = q\vec{E}$$

The remaining force, is the magnetic force which is seen from above equations to be

$$\vec{F}_{\text{mag}} = \vec{j}(-qvB_z) + \vec{k}(qvB_y)$$

Let $\vec{B} = \vec{i}B_x + \vec{j}B_y + \vec{k}B_z$ denote the magnetic field in frame S. The vector $\vec{v}$ is given by $\vec{v} = \vec{i}v + 0 + 0$. If we now calculate $\vec{v} \times \vec{B}$, we get

$\vec{v} \times \vec{B} = -\vec{j}vB_z + \vec{k}vB_y$. Hence the magnetic force may be written down as $q(\vec{v} \times \vec{B})$.

The total force on the charge q in frame S is then given by

$$\vec{F} = \vec{F_e} + \vec{F}_{\text{mag}} = q\vec{E} + q(\vec{v} \times \vec{B})$$

$$= q(\vec{E} + \vec{v} \times \vec{B})$$

Example 5 Show that in the limit of low speeds, the magnetic field due to a uniformly moving charge is given by Biot-Savart's law viz.

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q(\vec{v} \times \vec{r})}{r^3}$$

Solution The electric and magnetic fields due to a uniformly moving charge are given by

$$\vec{E} = \frac{q(1-\beta^2)}{4\pi\varepsilon_o r^3 (1-\beta^2 \sin^2\theta)^{3/2}}\vec{r}$$

and
$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2}$$

When the speed v is small, $v/c \ll 1$ and the electric field can be taken to be given by (neglecting $\beta^2 = v^2/c^2$)

$$\vec{E} \approx \frac{q\vec{r}}{4\pi\varepsilon_0 r^3}$$

Then
$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2} = \frac{1}{c^2}\frac{q(\vec{v} \times \vec{r})}{4\pi\varepsilon_o r^3}$$

Writing
$$c^2 = \frac{1}{\mu_0\varepsilon_0}, \text{ we get}$$

$$\vec{B} = \frac{\mu_0\varepsilon_0}{4\pi\varepsilon_0 r^3} q(\vec{v} \times \vec{r}) = \frac{\mu_0}{4\pi}\frac{q(\vec{v} \times \vec{r})}{r^3}$$

EXERCISES

1. What are (i) charge density (ii) current density? Obtain the equations for their transformation.
2. What do you mean by interdependence of electric and magnetic fields? Explain with a suitable example.
3. Two inertial frames S and S' move relative to one another with speed v. Derive the transformation equations relating the electric and magnetic fields in the two frames.

4. Write down the equations of transformations for electromagnetic fields in two inertial frames S and S' in relative motion. Use these to show that what is observed to be a pure electric or pure magnetic field in one frame is observed to have both electric as well as magnetic components in the other frame.
5. Derive an expression for the electric field generated by an electric charge in uniform motion relative to an observer. Explain how the field varies in the direction along and transverse to the motion of the charge.
6. An electric charge is in uniform motion relative to an inertial frame S. Obtain expressions for the electric and magnetic fields observed by an observer in frame S.
7. Explain why magnetism is said to be a relativistic phenomenon.
8. What do you mean by invariance of Maxwell's equations?
9. Show that the pair of equations $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ is invariant under L.T.
10. Show that the pair of electrons $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ and $\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \right)$ is invariant under L.T.
11. Using L.T. show that $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$.
12. Show that $c^2\rho^2 - (j_x^2 + j_y^2 + j_z^2) = c^2 \rho_0^2$.
13. Show that $(E^2 - c^2B^2)$ is relativistically invariant.
14. Show that $(\vec{E} \cdot \vec{B})$ is relativistically invariant.
15. Prove that $(E^2 - c^2B^2)$ is relativistically invariant. Hence show that if $E = cB$ in one inertial frame, then $E' = cB'$ in any other frame.
16. Prove that if the electric and magnetic fields are mutually perpendicular in one inertial frame, they must be mutually perpendicular in all inertial frames.

Suggested Further Reading

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